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Chapter 1

Introduction

1.1 What Is Virtual Reality?

Virtual reality (VR) technology is evolving rapidly, making it precarious to define VR in terms of specific devices that may fall out of favor in a year or two. In this book, we are concerned with fundamental principles that are less sensitive to particular technologies and therefore survive the test of time. Our first challenge is to consider what VR actually means, in a way that captures the most crucial aspects in spite of rapidly changing technology. The concept must also be general enough to encompass what VR is considered today and what we envision for its future.

We start with two representative examples that employ current technologies: 1) A human having an experience of flying over virtual San Francisco by flapping his own wings (Figure 1.1); 2) a mouse running on a freely rotating ball while exploring a virtual maze that appears on a projection screen around the mouse (Figure 1.2). We want our definition of VR to be broad enough to include these examples and many more, which are coming in Section 1.2. This motivates the following.

**Definition of VR:** Inducing targeted behavior in an organism by using artificial sensory stimulation, while the organism has little or no awareness of the interference.

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Figure 1.1: In the Birdly experience from the Zurich University of the Arts, the user, wearing a VR headset, flaps his wings while flying over virtual San Francisco, while a motion platform and fan provide additional sensory stimulation. The figure on the right shows the stimulus presented to each eye.

Figure 1.2: (a) An experimental setup used by neurobiologists at LMU Munich to present visual stimuli to rodents while they run on a spherical ball that acts as a treadmill (Figure by Kay Thurley). (b) A picture of a similar experiment, performed at Princeton University.
1.1. WHAT IS VIRTUAL REALITY?

Four key components appear in the definition:

- **Targeted behavior:** The organism is having an “experience” that was designed by the creator. Examples include flying, walking, exploring, watching a movie, and socializing with other organisms.

- **Organism:** This could be you, someone else, or even another life form such as a fruit fly, cockroach, fish, rodent, or monkey (scientists have used VR on all of these!).

- **Artificial sensory stimulation:** Through the power of engineering, one of more senses of the organism become hijacked, and their ordinary inputs are replaced by artificial stimulation.

- **Awareness:** While having the experience, the organism seems unaware of the interference, thereby being “fooled” into feeling present in a virtual world. The unawareness may lead to a sense of presence in another world, or acceptance of it being natural.

Who is the fool? The idea of “fooling” an organism might seem fluffy or meaningless; however, this can be made surprisingly concrete using research from neurobiology. When animals explore their environment, neural structures composed of place cells are formed that encode spatial information about their surroundings [38]; see Figure 1.3(a). Each place cell is activated precisely when the organism returns to a particular location that is covered by it. Although less understood, grid cells even encode locations in a manner similar to Cartesian coordinates [36] (Figure 1.3(b)). It has been shown that these neural structures may form in an organism, even when having a VR experience [7, 19]. In other words, our brains may form place cells for places that are not real! This is a clear indication that VR is fooling our brains, at least partially.

**Terminology regarding various “worlds”** Several terms related to VR are in common use at present. The term virtual environments predates widespread usage of VR, and is commonly considered to be synonymous; however, we emphasize in this book that VR can be an interface to a captured “real” world just as well as experiencing a completely artificial world. Augmented reality (AR) refers to systems in which most of the visual stimuli are propagated directly through glass or cameras to the eyes, and some additional structures appear to be superimposed onto the user’s world. The term mixed reality is sometimes used to refer to an entire spectrum that encompasses VR, AR, and normal reality. Telepresence refers to systems that enable users to feel like they are somewhere else in the real world; if they are able to control anything, such as a flying drone, then teleoperation is an appropriate term. For our purposes, virtual environments, AR, mixed reality, telepresence, and teleoperation will all be considered as perfect examples of VR.

The most important idea of VR is that the user’s perception of reality has been altered through engineering, rather than whether the environment they believe they are in seems more “real” or “virtual”. We will instead use these terms to distinguish whether VR is employed: The real world refers to the physical world that contains the user, and the virtual world refers to the perceived world as part of the targeted VR experience.

**Interactivity** Most VR experiences involve another crucial component: interaction. Does the sensory stimulation depend on actions taken by the organism? If the answer is “no”, then the VR system is called open-loop; otherwise, it is closed-loop. In the case of closed-loop VR, the organism has partial control over the stimulation, which could vary as a result of body motions, including eyes, head, hands, or legs. Other possibilities include voice commands, heart rate, body temperature, and skin conductance (are you sweating?).

**First- vs. Third-person** If you are reading this book, then you most likely want to develop VR systems or experiences. Pay close attention to this next point! When a scientist designs an experiment for an organism, as shown in Figure 1.2, then the separation is clear: The laboratory subject (organism) has a first-person experience, while the scientist is a third-person observer. The scientist carefully designs the VR system as part of an experiment that will help to resolve a scientific hypothesis. For example, does turning off a few neurons in a rat’s brain affect its navigation ability? On the other hand, when engineers or developers construct a VR system or experience, they are usually targeting themselves and people like them. They feel perfectly comfortable moving back and forth between being
1.1. WHAT IS VIRTUAL REALITY?

The ‘scientist’ and the “lab subject” while evaluating and refining their work. As you will learn throughout this book, this is a bad idea! The creators of the experience are heavily biased by their desire for it to succeed without having to redo their work. They also know what the experience is supposed to mean or accomplish, which provides a strong bias in comparison to a fresh subject. To complicate matters further, the creator’s body will physically and mentally adapt to whatever flaws are present so that they may soon become invisible. We have seen these kinds of things before. For example, it is hard to predict how others will react to your own writing. Also, it is usually harder to proofread your own writing in comparison to that of others. In the case of VR, these effects are much stronger and yet elusive to the point that you must force yourself to pay attention to them. Take great care when hijacking the senses that you have trusted all of your life. This will most likely be uncharted territory for you.

More real than reality? How “real” should the VR experience be? It is tempting to try to make it match our physical world as closely as possible. Our brains are most familiar with this setting, thereby making it seem most appropriate. This philosophy has dominated the video game industry at times, for example, in the development of highly realistic first-person-shooter (FPS) games that are beautifully rendered on increasingly advanced graphics cards. In spite of this, understand that extremely simple, cartoon-like environments can also be effective and even preferable. Examples appear throughout history, as discussed in Section 1.3.

As a VR experience creator, think carefully about the task, goals, or desired effect you want to have on the user. You have the opportunity to make the experience “better than real”. What will they be doing? Taking a math course? Experiencing a live theatrical performance? Writing software? Designing a house? Maintaining a long-distance relationship? Playing a game? Having a meditation and relaxation session? Traveling to another place on Earth, or in the universe? For each of these, think about how the realism requirements might vary. For example, consider writing software in VR We currently write software by typing into windows that appear on a large screen. Note that even though this is a familiar experience for many people, it was not even possible in the physical world of the 1950s. In VR, we could simulate the modern software development environment by convincing the programmer that she is sitting in front of a screen; however, this misses the point that we can create almost anything in VR. Perhaps a completely new interface will emerge that does not appear to be a screen sitting on a desk in an office. For example, the windows could be floating above a secluded beach or forest. Furthermore, imagine how a debugger could show the program execution trace.

Synthetic vs. captured Two extremes exist when constructing a virtual world as part of a VR experience. At one end, we may program a synthetic world, which is completely invented from geometric primitives and simulated physics. This is common in video games and such virtual environments were assumed to be the main way to experience VR in earlier decades. At the other end, the world may be captured using modern imaging techniques. For viewing on a screen, the video camera has served this purpose for over a century. Capturing panoramic images and videos and then seeing them from any viewpoint in a VR system is a natural extension. In many settings, however, too much information is lost when projecting the real world onto the camera sensor. What happens when the user changes her head position and viewpoint? More information should be captured in this case. Using depth sensors and SLAM (Simultaneous Localization And Mapping) techniques, a 3D representation of the surrounding world can be captured and maintained over time as it changes. It is extremely difficult, however, to construct an accurate and reliable representation, unless the environment is explicitly engineered for such capture (for example, a motion capture studio).

As humans interact, it becomes important to track their motions, which is an important form of capture. What are their facial expressions while wearing a VR headset? Do we need to know their hand gestures? What can we infer about their emotional state? Are their eyes focused on me? Synthetic representations of ourselves called avatars enable us to interact and provide a level of anonymity, if desired in some contexts. The attentiveness or emotional state can be generated synthetically. We can also enhance our avatars by tracking the motions and other attributes of our actual bodies. A well-known problem is the uncanny valley, in which a high degree of realism has been achieved in a avatar, but its appearance makes people feel uneasy. It seems almost right, but the small differences are disturbing. There is currently no easy way to make ourselves appear to others in a VR experience exactly as we do in the real world, and in most cases, we might not want to.

Health and safety Although the degree of required realism may vary based on the tasks, one requirement remains invariant: The health and safety of the users. Unlike simpler media such as radio or television, VR has the power to overwhelm the senses and the brain, leading to fatigue or sickness. This phenomenon has been studied under the heading simulator sickness for decades; in this book we will refer to adverse symptoms from VR usage as VR sickness. Sometimes the discomfort is due to problems in the VR hardware and low-level software; however, in most cases, it is caused by a careless VR developer who misunderstands or disregards the side effects of the experience on the user. This is one reason why human physiology and perceptual psychology are large components of this book. To develop comfortable VR experiences, you must understand how these factors interact. In many cases, fatigue arises because the brain appears to work harder to integrate the unusual stimuli being presented to the senses. In some cases, inconsistencies with prior expectations, and outputs from other senses, even lead to dizziness and nausea.

Another factor that leads to fatigue is an interface that requires large amounts of muscular effort. For example, it might be tempting move objects around in a sandbox game by moving your arms around in space. This quickly leads to
fatigue and an avoidable phenomenon called *gorilla arms*, in which people feel that the weight of their extended arms is unbearable. For example, by following the principle of the computer mouse, it may be possible to execute large, effective motions in the virtual space by small, comfortable motions of a controller. Over long periods of time, the brain will associate the motions well enough for it to seem realistic while also greatly reducing fatigue.

### 1.2 Modern VR Experiences

This section gives you a quick overview of what people are doing with VR today, and provides a starting point for searching for similar experiences on the Internet. Here, we can only describe the experiences in words and pictures, which is a long way from the appreciation gained by experiencing them yourself. This printed medium (a book) is woefully inadequate for fully conveying the medium of VR. Perhaps this is how it was in the 1890s to explain in a newspaper what a movie theater was like! If possible, it is strongly recommended that you try many VR experiences yourself to form first-hand opinions and spark your imagination to do something better.

**Video games** People have dreamed of entering their video game worlds for decades. By 1982, this concept was already popularized by the Disney movie Tron. Figure 1.4 shows several video game experiences in VR. Most gamers currently want to explore large, realistic worlds through an avatar. Figure 1.4(a) shows Valve’s Portal 2, which is a puzzle-solving adventure game developed for the HTC Vive VR headset. Figure 1.4(b) shows an omnidirectional treadmill peripheral that gives users the sense of walking while they slide their feet in a dish on the floor. These two examples give the user a *first-person* perspective of their character. By contrast, Figure 1.4(c) shows Lucky’s Tale, which instead yields a comfortable *third-person* perspective as the player floats above his character. Figure 1.4(d) shows a game that contrasts all the others in that it was designed to specifically exploit the power of VR.

**Immersive cinema** Hollywood movies continue to offer increasing degrees of realism. Why not make the viewers feel like they are part of the scene? Figure 1.5 shows an immersive short story. Movie directors are entering a fascinating new era of film. The tricks of the trade that were learned across the 20th century need to be reinvestigated because they are based on the assumption that the cinematographer controls the camera viewpoint. In VR, viewers can look in any direction, and perhaps even walk through the scene. What should they be allowed to do? How do you make sure they do not miss part of the story? Should the story be linear, or should it adapt to the viewer’s actions? Should the viewer be a first-person character in the film, or a third-person observer who in invisible to the other characters? How can a group of friends experience a VR film together? When are animations more appropriate versus the capture of real scenes?

It will take many years to resolve these questions and countless more that will arise. In the meantime, VR can also be used as a kind of “wrapper” around existing movies. Figure 1.6 shows the VR Cinema application, which allows the user to choose any seat in a virtual movie theater. Whatever standard movies or videos that are on the user’s hard drive can be streamed to the screen in the theater. These could be 2D or 3D movies. A projector in the back emits flickering lights and the audio is adjusted to mimic the acoustics of a real theater. This provides an immediate way to leverage all content that was developed for viewing on a screen, and bring it into VR. Many simple extensions can be made without modifying the films. For example, in a movie about zombies, a few virtual zombies could enter the theater and start to chase you. In a movie about tornadoes, perhaps the theater rips apart. You can also have a social experience. Imagine having “movie night” with your friends from around the world, while you sit together in the virtual movie theater. You can even have the thrill of misbehaving in the theater without getting thrown out.
1.2. MODERN VR EXPERIENCES

Figure 1.5: Oculus Story Studio produced *Henry*, an immersive short story about an unloved hedgehog who hopes to make a new friend, the viewer.

Figure 1.6: VR Cinema, developed by Joo-Hyung Ahn for the Oculus Rift. You can choose your seat and watch any movie you like.

Figure 1.7: An important component for achieving telepresence is to capture a panoramic view: (a) A car with cameras and depth sensors on top, used by Google to make Street View. (b) Bublcam is a cheap, portable way to capture and stream omnidirectional videos.

Telepresence The first step toward feeling like we are somewhere else is capturing a panoramic view of the remote environment (Figure 1.7). Google’s Street View and Earth apps already rely on the captured panoramic images from millions of locations around the world. Simple VR apps that query the Street View server directly enable the user to feel like he is standing in each of these locations, while easily being able to transition between nearby locations (Figure 1.8). Panoramic video capture is even more compelling. Figure 1.9 shows a frame from an immersive rock concert experience. Even better is to provide live panoramic video interfaces, through which people can attend sporting events and concerts. Through a live interface, interaction is possible. People can take video conferencing to the next level by feeling present at the remote location. By connecting panoramic cameras to robots, the user is even allowed to move around in the remote environment (Figure 1.10). Current VR technology allows us to virtually visit far away places and interact in most of the ways that were previously possible only while physically present. This leads to improved opportunities for telecommuting to work. This could ultimately help reverse the urbanization trend sparked by the 19th-century industrial revolution, leading to *deurbanization* as we distribute more uniformly around the Earth.

Virtual societies Whereas telepresence makes us feel like we are in another part of the physical world, VR also allows us to form entire societies that remind us of the physical world, but are synthetic worlds that contain avatars connected to real people. Figure 1.11 shows a Second Life scene in which people interact in a fantasy world through avatars; such experiences were originally designed to view on a screen but can now be experienced through VR. Groups of people could spend time together in these spaces for a variety of reasons, including common
1.2. MODERN VR EXPERIENCES

Figure 1.8: A simple VR experience that connects resents Google Street View images through a VR headset: (a) A familiar scene in Paris. (b) Left and right eye views are created inside the headset, while also taking into account the user’s looking direction.

Figure 1.9: Jaunt captured a panoramic video of Paul McCartney performing Live and Let Die, which provides a VR experience where users felt like they were on stage with the rock star.

Figure 1.10: Examples of robotic avatars: (a) The DORA robot from the University of Pennsylvania mimics the users head motions, allowing him to look around in a remote world while maintaining a stereo view (panoramas are monoscopic). (b) The Plexidrone, a low-cost flying robot that is designed for streaming panoramic video.

Figure 1.11: Virtual societies develop through interacting avatars that meet in virtual worlds that are maintained on a common server. A snapshot from Second Life is shown here.
special interests, educational goals, or simple an escape from ordinary life.

Empathy  The first-person perspective provided by VR is a powerful tool for causing people to feel *empathy* for someone else’s situation. The world continues to struggle with acceptance and equality for others of different race, religion, age, gender, sexuality, social status, and education, while the greatest barrier to progress is that most people cannot fathom what it is like to have a different identity. Figure 1.12 shows a VR project sponsored by the United Nations to yield feelings of empathy for those caught up in the Syrian crisis of 2015. Some of us may have compassion for the plight of others, but it is a much stronger feeling to understand their struggle because you have been there before. Figure 1.13 shows a VR system that allows men and women to swap bodies. Through virtual societies, many more possibilities can be explored. What if you were 10cm shorter than everyone else? What if you teach your course with a different gender? What if you were the victim of racial discrimination by the police? Using VR, we can imagine many “games of life” where you might not get as far without being in the “proper” group.

Education  In addition to teaching empathy, the first-person perspective could revolutionize many areas of education. In engineering, mathematics, and the sciences, VR offers the chance to visualize geometric relationships in difficult concepts or data that is hard to interpret. Furthermore, VR is naturally suited for practical training because skills developed in a realistic virtual environment may transfer naturally to the real environment. The motivation is particularly high if the real environment is costly to provide or poses health risks. One of the earliest and most common examples of training is VR is *flight simulation* (Figure 1.14). Other examples include firefighting, nuclear power plant safety, search-and-rescue, military operations, and medical procedures.

Beyond these common uses of VR, perhaps the greatest opportunities for VR education lie in the humanities, including history, anthropology, and foreign language acquisition. Consider the difference between reading a book on the Victorian era in England and being able to roam the streets of 19th-century London, in a simulation that has been painstakingly constructed by historians. We could even visit an ancient city that has been reconstructed from ruins (Figure 1.15). Fascinating possibilities exist for either touring *physical* museums through a VR interface or scanning and exhibiting artifacts directly in *virtual* museums.

Virtual prototyping  In the real world, we build prototypes to understand how a proposed design feels or functions. Thanks to 3D printing and related technologies, this is easier than ever. At the same time, *virtual prototyping* enables designers to inhabit a virtual world that contains their prototype (Figure 1.16). They can quickly interact with it and make modifications. They also have the opportunities to bring clients into their virtual world so that they can communicate
1.2 MODERN VR EXPERIENCES

Figure 1.14: A flight simulator used by the US Air Force (photo by Javier Garcia). The user sits in a physical cockpit while being surrounded by displays that show the environment.

Figure 1.15: A tour of the Nimrud palace of Assyrian King Ashurnasirpal II, a VR experience developed by Learning Sites Inc. and the University of Illinois.

Figure 1.16: Architecture is a prime example of where a virtual prototype is invaluable. This demo, called Ty Hedfan, was created by designer Olivier Demangel. The real kitchen is above and the virtual kitchen is below.
1.2 MODERN VR EXPERIENCES

Figure 1.17: A heart visualization system based on images of a real human heart. This was developed by the Jump Trading Simulation and Education Center and the University of Illinois.

their ideas. Imagine you want to remodel your kitchen. You could construct a model in VR and then explain to a contractor exactly how it should look. Virtual prototyping in VR has important uses in many businesses, including real estate, architecture, and the design of aircraft, spacecraft, cars, furniture, clothing, and medical instruments.

Health care Although health and safety are challenging VR issues, the technology can also help to improve our health. There is an increasing trend toward distributed medicine, in which doctors train people to perform routine medical procedures in remote communities around the world. Doctors can provide guidance through telepresence, and also use VR technology for training. In another use of VR, doctors can immerse themselves in 3D organ models that were generated from medical scan data (Figure 1.17). This enables them to better plan and prepare for a medical procedure by studying the patient’s body shortly before an operation. They can also explain medical options to the patient or his family so that they may make more informed decisions. In yet another use, VR can directly provide therapy to help patients. Examples include overcoming phobias and stress disorders through repeated exposure, improving or maintaining cognitive skills in spite of aging, and improving motor skills to overcome balance, muscular, or nervous system disorders.

New human experiences Finally, the point might be to simply provide a new human experience. Through telepresence, people can try experiences through the eyes of robots or other people. However, we can go further by giving people experiences that are impossible (or perhaps deadly) in the real world. Most often, artists are the ones leading this effort. The Birdly experience of human flying (Figure 1.1) was an excellent example. Figure 1.18 shows two more. What if we change our scale? Imagine being 2mm tall and looking ants right in the face. Compare that to being 50m tall and towering over a city while people scream and run from you. What if we simulate the effect of drugs in your system? What if you could become your favorite animal? What if you became a piece of food? The creative possibilities for artists seem to be endless. We are limited only by what our bodies can comfortably handle. Exciting adventures lie ahead!

1.3 History Repeats

Staring at rectangles How did we arrive to VR as it exists today? We start with a history that predates what most people would consider to be VR, but includes many aspects crucial to VR that have been among us for tens of thousands of years. Long ago, our ancestors were trained to look at the walls and imagine a 3D world that is part of a story. Figure 1.19 shows some examples of this. Cave paintings, such as the one shown in Figure 1.19(a) from 30,000 years ago. Figure 1.19(b) shows a painting from the European Middle Ages. Similar to the cave painting, it relates to military conflict, a fascination of humans regardless of the era or technology. There is much greater detail in the newer painting, leaving
1.3. HISTORY REPEATS

Flipping quickly through a sequence of pictures gives the illusion of motion, even at a rate as low as two pictures per second. Above ten pictures per second, the motion even appears to be continuous, rather than perceived as individual pictures. One of the earliest examples of this effect is the race horse movie created by Eadward Muybridge in 1878 at the request of Leland Stanford (yes, that one!); see Figure 1.20.

Motion picture technology quickly improved, and by 1896, a room full of spectators in a movie theater screamed in terror as a short film of a train pulling into a station convinced them that the train was about to crash into them (Figure 1.21(a)). There was no audio track. Such a reaction seems ridiculous for anyone who has been to a modern movie theater. As audience expectations increased, so has the degree of realism produced by special effects. In 1902, viewers were inspired by a Journey to the Moon (Figure 1.21(b)), but by 2013, an extremely high degree of realism seemed necessary to keep viewers believing (Figure 1.21(c) and 1.21(d)).

At the same time, motion picture audiences have been willing to accept lower degrees of realism. One motivation, as for paintings, is to leave more to the imagination. The popularity of animation (also called anime or cartoons) is a prime example (Figure 1.22). Even within the realm of animations, a similar trend has emerged as with motion pictures in general. Starting from simple line drawings in 1908 with Fantasmagorie (Figure 1.22(a)), greater detail appears in 1928 with the introduction of Mickey Mouse(Figure 1.22(b)). By 2003, animated films achieved a much higher degree of realism (Figure 1.22(c)); however, excessively simple animations have also enjoyed widespread popularity (Figure 1.22(d)).

Moving pictures Once humans were content with staring at rectangles on the wall, the next step was to put them into motion. The phenomenon of stroboscopic apparent motion is the basis for what we call movies or motion pictures today.

Figure 1.19: (a) A 30,000-year-old painting from the Bhimbetka rock shelters in India (photo by Archaeological Survey of India). (b) An English painting from around 1470 that depicts John Ball encouraging Wat Tyler rebels (unknown artist). (c) A painting by Hans Vredeman de Vries in 1596. (d) An impressionist painting by Claude Monet in 1874.

less to the imagination; however, the drawing perspective is comically wrong. Some people seem short relative to others, rather than being further away. The rear portion of the fence looks incorrect. Figure 1.19(c) shows a later painting in which the perspective have been meticulously accounted for, leading to a beautiful palace view that requires no imagination for us to perceive it as "3D". By the 19th century, many artists had grown tired of such realism and started the controversial impressionist movement, an example of which is shown in Figure 1.19(d). Such paintings leave more to the imagination of the viewer, much like the earlier cave paintings.

Figure 1.20: This 1878 motion picture by Eadward Muybridge, was created by evenly spacing 24 cameras along a track and triggering them by trip wire as the horse passes. The animation was played on a zoopraxiscope, which was a precursor to the movie projector, but was mechanically similar to a record player.
1.3. HISTORY REPEATS

Figure 1.21: (a) Arrival of a Train at La Ciotat Station, 1896. (b) A Trip to the Moon, 1902. (c) The movie 2001, from 1968. (d) Gravity, 2013.

Toward convenience and portability Another motivation for accepting lower levels of realism is cost and portability. As shown in Figure 1.23, families were willing to gather in front of a television to watch free broadcasts in their homes, even though they could go to theaters and watch high-resolution, color, panoramic, and 3D movies at the time. Such tiny, blurry, black-and-white television sets seem comically intolerable with respect to our current expectations. The next level of portability is to carry the system around with you. Thus, the progression is from: 1) having to go somewhere to watch it, to 2) being able to watch it in your home, to 3) being able to carry it anywhere. Whether pictures, movies, phones, computers, or video games, the same progression continues. We can therefore expect the same for VR systems. At the same time, note that the gap is closing between these levels: The quality we expect from a portable device is closer than ever before to the version that requires going somewhere to experience it.

Video games Motion pictures yield a passive, third-person experience, in contrast to video games which are closer to a first-person experience by allowing us to interact with him. Recall from Section 1.1 the differences between open-loop and closed-loop VR. Video games are an important step closer to closed-loop VR, whereas motion pictures are open-loop. As shown in Figure 1.24, we see the same trend from simplicity to improved realism and then back to simplicity. The earliest games, such as Pong and Donkey Kong, left much to the imagination. First-person shooter (FPS) games such as Doom gave the player a first-person perspective and launched a major campaign over the following decade toward higher quality graphics and realism. Assassin’s Creed shows a typical scene from a modern, realistic video game. At the same time, wildly popular games have emerged by focusing on simplicity. Angry Birds looks reminiscent of games from the 1980s, and Minecraft allows users to create and inhabit worlds composed of course blocks. Note that reduced realism often leads to simpler engineering requirements; in 2015, an advanced FPS game might require a powerful PC and graphics card, while simpler games would run on a basic smartphone. Repeated lesson: Don’t assume that more realistic is better!

Beyond staring at a rectangle The concepts so far are still closely centered on staring at a rectangle that is fixed on a wall. Two important steps come next:
1.3. HISTORY REPEATS

Figure 1.23: Although movie theaters with large screens were available, families were also content to gather around television sets that produced a viewing quality that would be unbearable by current standards, as shown in this photo from 1958.

1) Presenting a separate picture to each eye to induce a “3D” effect. 2) Increasing the field of view so that the user is not distracted by anything but the stimulus. One way our brains infer the distance of objects from our eyes is by stereopsis. Information is gained by observing and matching features in the world that are visible to both the left and right eyes. The differences between their images on the retina yield cues about distances; keep in mind that there are many more such cues, which we discuss in Section 6.1. The first experiment that showed this 3D effect of stereopsis was performed in 1838 by Charles Wheatstone in a system called the stereoscope (Figure 1.25(a)). By the 1930s, a portable version became a successful commercial product known to this day as the View-Master (Figure 1.25(b)). Pursuing this idea further led to Sensorama, which added motion pictures, sound, vibration, and even smells to the experience (Figure 1.25(c)). An unfortunate limitation of these designs is requiring that the viewpoint is fixed with respect to the picture. If the device is too large, then the user’s head also becomes fixed in the world. An alternative has been available in movie theaters since the 1950s. Stereopsis is achieved when participants wore special glasses that select a different image for each eye using polarized light filters. This popularized 3D movies, which are viewed the same way in the theaters today.

1.3. HISTORY REPEATS

Another way to increase the sense of immersion and depth is to increase the field of view. The Cinerama system from the 1950s offered a curved, wide field of view that is similar to the curved, large LED displays offered today (Figure 1.25(d)). Along these lines, we could place screens all around us. This idea led to one important family of VR systems called the CAVE, which was introduced in 1992 at the University of Illinois [9] (Figure 1.26(a)). The user enters a room in which video is projected onto several walls. The CAVE system also offers stereoscopic viewing by presenting different images to each eye using polarized light and special glasses. Often, head tracking is additionally performed to allow viewpoint-dependent video to appear on the walls.

VR headsets Once again, the trend toward portability appears. An important step for VR was taken in 1968 with the introduction of Ivan Sutherland’s Sword of Damocles, which leveraged the power of modern displays and computers (Figure 1.26(b)). He constructed what is widely considered to be the first VR headset. As the user turns his head, the images presented on the screen are adjusted to compensate so that the virtual objects appear to be fixed in space. This yielded the first glimpse of an important concept in this book: The perception of stationarity. To make an object appear to be stationary while you move your sense organ, the device producing the stimulus must change its output to compensate for the motion. This requires sensors and tracking systems to become part of the VR system. Commercial VR headsets started appearing in the 1980s with Jaron Lanier’s company VPL, thereby popularizing the image of goggles and gloves; Figure 1.26(c). In the 1990s, VR-based video games appeared in arcades (Figure 1.26(d)) and at home units (Figure 1.26(e). The experiences were not compelling or comfortable enough to attract mass interest. However, the current generation of VR headset leverages the widespread availability of high resolution screens and sensors, due to the smartphone industry, to offer lightweight, low-cost, high-field-of-view headsets, such as the Oculus Rift (Figure 1.26(f)). This has greatly improved the quality of VR experiences while significantly lowering the barrier of entry for developers and hobbyists. This also caused a recent flood of interest in VR technology and applications.

Bringing people together We have so far neglected an important aspect, which is human-to-human or social interaction. We use formats such as a live theater performance, a classroom, or a lecture hall for a few people to communicate with or entertain a large audience. We write and read novels to tell stories to each other. Prior to writing, skilled storytellers would propagate experiences to others, including future generations. We have communicated for centuries by writing letters to each other. More recent technologies have allowed us to interact directly without delay. The audio part has been transmitted through telephones for over a century, and now the video part is transmitted as well through videoconferencing over the Internet. At the same time, simple text messaging has become a valuable part of our interaction, providing yet another example of a preference.

Figure 1.25: (a) The first stereoscope, developed by Wheatstone in 1838, used mirrors to present a different image to each eye; the mirrors were replaced by lenses soon afterward. (b) The View-Master is a mass-produced stereoscope that has been available since the 1930s. (c) In 1957, Sensorama added motion pictures, sound, vibration, and even smells to the experience. (d) In competition to stereoscopic viewing, Cinerama offered a larger field of view. Larger movie screens caused the popularity of 3D movies to wane in the 1950s.
1.3. HISTORY REPEATS

Figure 1.26: (a) CAVE VR, 1992. (b) Sword of Damocles, 1968. (c) VPL Eye-phones, 1980s. (d) Virtuality gaming, 1990s. (e) Nintendo Virtual Boy, 1995. (f) Oculus Rift DK2, 2014.

Figure 1.27: Second Life was introduced in 2003 as a way for people to socialize through avatars and essentially build a virtual world to live in. Shown here is the author giving a keynote address at the 2014 OpenSimulator Community Conference. The developers build open source software tools for constructing and hosting such communities of avatars with real people behind them.

for decreased realism. Communities of online users who interact through text messages over the Internet have been growing since the 1970s. In the context of games, early Multi-User Dungeons (MUDs) grew into Massively Multiplayer Online Games (MMORPGs) that we have today. In the context of education, the PLATO system from the University of Illinois was the first computer-assisted instruction system, which included message boards, instant messaging, screen sharing, chat rooms, and emoticons. This was a precursor to many community-based, online learning systems, such as the Kahn Academy and Coursera. The largest amount of online social interaction today occurs through Facebook apps, which involves direct communication through text along with the sharing of pictures, videos, and links.

Returning to VR, we can create avatar representations of ourselves and “live” together in virtual environments, as is the case with Second Life and OpenSimulator 1.27. Without being limited to staring at rectangles, what kinds of societies will emerge with VR? Popular science fiction novels have painted a thrilling, yet dystopian future of a world where everyone prefers to interact through VR [8, 17, 47]. It remains to be seen what the future will bring.

As the technologies evolve over the years, keep in mind the power of simplicity when making a VR experience. In some cases, maximum realism may be important; however, leaving much to the imagination of the users is also valuable. Although the technology changes, one important invariant is that humans are still designed the same way. Understanding how our senses, brains, and bodies work is crucial to understanding the fundamentals of VR systems.
Further reading


Chapter 2

Bird’s-Eye View

This chapter presents an overview of VR systems from hardware (Section 2.1 to software 2.2 to human perception 2.3). The purpose is to quickly provide a complete perspective so that the detailed subjects in the remaining chapters are understood within the larger context.

2.1 Hardware

The first step to understanding how VR works is to consider what constitutes the entire VR system. It is tempting to think of it as being merely the hardware components, such as computers, headsets, and controllers. This would be woefully incomplete. As shown in Figure 2.1, it is equally important to account for the organism, which in this chapter will exclusively refer to a human user. The VR hardware produces stimuli that override the senses of the user. In the Sword of Damocles from Section 1.3 (Figure 1.26(b)), recall that tracking is needed to adjust the stimulus based on human motions. The VR hardware accomplishes this by using its own sensors, thereby tracking motions of the user. Head tracking is the most important, but tracking also may include button presses, controller movements, eye movements, or the movements of any other body parts. Finally, it is also important to consider the surrounding physical world as part of the VR system. In spite of stimulation provided by the VR hardware, the user will always have other senses that respond to stimuli from the real world. She also has the ability to change her environment through body motions. The VR hardware might also track objects other than the user, especially if interaction with them is part of the VR experience. Through a robotic interface, the VR hardware might also change the real world. One example is teleoperation of a robot through a VR interface.

Sensors and sense organs How is information extracted from the physical world? Clearly this is crucial to a VR system. In engineering, a transducer refers to a device that converts energy from one form to another. A sensor is a special transducer that converts the energy it receives into a signal for an electrical circuit. This may be an analog or digital signal, depending on the circuit type. A sensor typically has a receptor that collects the energy for conversion. Organisms work in a similar way. The “sensor” is called a sense organ, with common examples being eyes and ears. Because our “circuits” are formed from interconnected neurons, the sense organs convert energy into neural impulses. As you progress through this book, keep in mind the similarities between engineered sensors and natural sense organs. They are measuring the same things and sometimes even function in a similar manner. This should not be surprising because we and our engineered devices share the same physical world: The laws of physics and chemistry remain the same.

Configuration space of sense organs As the user moves through the physical world, his sense organs move along with him. Furthermore, some sense organs move relative to the body skeleton, such as our eyes rotating within their sockets. Each sense organ has a configuration space, which corresponds to all possible ways it can be transformed or configured. The most important aspect of this is the
2.1. HARDWARE

Figure 2.2: Under normal conditions, the brain (and body parts) control the configuration of sense organs (eyes, ears, fingertips) as they receive natural stimulation from the surrounding, physical world.

Figure 2.3: In comparison to Figure 2.2, a VR system “hijacks” each sense by replacing the natural stimulation with artificial stimulation that is provided by hardware called a display. Using a computer, a virtual world generator maintains a coherent, virtual world. Appropriate “views” of this virtual world are rendered to the display.

Figure 2.4: If done well, the brain is “fooled” into believing that the virtual world is in fact the surrounding physical world and natural stimulation is resulting from it.

number of degrees of freedom or DOFs of the sense organ. Chapter 3 will cover this thoroughly, but for now note that a rigid object that moves through ordinary space has six DOFs. Three DOFs correspond to its changing position in space: 1) side-to-side motion, 2) vertical motion, and 3) closer-further motion. The other three DOFs correspond to possible ways the object could be rotated; in other words, exactly three independent parameters are needed to specify how the object is oriented. These are called yaw, pitch, and roll, and are covered in Section 3.2.

As an example, consider your left ear. As you rotate your head or move your body through space, the position of the ear changes, as well as its orientation. This yields six DOFs. The same is true for your right eye. Keep in mind that our bodies have many more degrees of freedom, which affect the configuration of our sense organs. A tracking system may be necessary to determine the position and orientation of each sense organ that receives artificial stimuli, which will be explained shortly.
An abstract view  Figure 2.2 illustrates the normal operation of one of our sense organs without interference from VR hardware. The brain controls its configuration, while the sense organ converts natural stimulation from the environment into neural impulses that are sent to the brain. Figure 2.3 shows how it appears in a VR system. The VR hardware contains several components that will be discussed shortly. A Virtual World Generator (VWG) runs on a computer and produces “another world”, which could be many possibilities, such as a pure simulation of a synthetic world, a recording of the real world, or a live connection to another part of the real world. The human perceives the virtual world through each targeted sense organ using a display, which emits energy that is specifically designed to mimic the type of stimulus that would appear without VR. The process of converting information from the VWG into output for the display is called rendering.

In the case of human eyes, the display might be a smartphone screen or the screen of a video projector. In the case of ears, the display is referred to as a speaker. (A display need not be visual, even though this is the common usage in everyday life.) If the VR system is effective, then the brain is hopefully “fooled” in the sense shown in Figure 2.4. The user should believe that the stimulation of the senses is natural and comes from a plausible world, being consistent with at least some past experiences.

Aural: world-fixed vs. user-fixed  Recall from Section 1.3 the trend of having to go to somewhere for an experience, to having it in the home, and then finally to having it be completely portable. To understand these choices for VR systems and their implications on technology, it will be helpful to compare a simpler case: Audio or aural systems.

Figure 2.5 shows the speaker setup and listener location for a Dolby 7.1 Surround Sound theater system, which could be installed at a theater or a home family room. Seven speakers distributed around the room periphery generate most of the sound, while a subwoofer (the “1” of the “7.1”) delivers the lowest frequency components. The aural displays are therefore world-fixed. Compare this to a listener wearing headphones, as shown in Figure 2.6. In this case, the aural displays are user-fixed. Hopefully, you have already experienced settings similar to these many times.

What are the key differences? In addition to the obvious portability of headphones the following quickly come to mind:

- In the surround-sound system, the generated sound (or stimulus) is far away from the ears, whereas it is quite close for the headphones.
- One implication of the difference in distance is that much less power is needed for the headphones to generate an equivalent perceived loudness level compared with distant speakers.
- Another implication based on distance is the degree of privacy allowed by the wearer of headphones. A surround-sound system at high volume levels could generate a visit by angry neighbors.
Wearing electronics on your head could be uncomfortable over long periods of time, causing a preference for surround sound over headphones.

Several people can enjoy the same experience in a surround-sound system (although they cannot all sit in the optimal location). Using headphones, they would need to split the audio source across their individual headphones simultaneously.

They are likely to have different costs, depending on the manufacturing costs and available component technology. At present, headphones are favored by costing much less than a set of surround-sound speakers (although one can spend large amount of money on either).

All of these differences carry over to VR systems. This should not be too surprising because we could easily consider a pure audio experience to be a special kind of VR experience based on our definition from Section 1.1.

While listening to music, close your eyes and imagine you are at a live performance with the artists surrounding you. Where do you perceive the artists and their instruments to be located? Are they surrounding you, or do they seem to be in the middle of your head? Using headphones, it is most likely that they seem to be inside your head. In a surround-sound system, if recorded and displayed properly, the sounds should seem to be coming from their original locations well outside of your head. They probably seem constrained, however, into the horizontal plane that you are sitting in.

This shortcoming of headphones is not widely recognized at present, but nevertheless represents a problem that becomes much larger for VR systems that include visual displays. If you want to preserve your perception of where sounds are coming from, then headphones would need to take into account the configuration of your ears in space so that audio is adjusted accordingly. For example, if you nod your head back and forth in a “no” gesture, then the sound being presented to each ear needs to be adjusted so that the simulated sound source is rotated in the opposite direction. In the surround-sound system, the speaker does not follow your head and therefore does not need to rotate. If the speaker rotates with your head, then a counter-rotation is needed to “undo” your head rotation so that the sound source location is perceived to be stationary.

**Visual: world-fixed vs. user-fixed**

Now consider adding a visual display. You might not worry much about the perceived location of artists and instruments while listening to music, but you will quickly notice if their locations do not appear correct to your eyes. Our vision sense is much more powerful and complex than our sense of hearing. Figure 2.7(a) shows a CAVE system, which parallels the surround-sound system in many ways. The user again sits in the center while displays around the periphery present visual stimuli to your eyes. The speakers are replaced by video screens. Figure 2.7(b) shows a user wearing a VR headset, which parallels the headphones.

Figure 2.7: (a) A CAVE VR system developed at Teesside University, UK. (b) A woman wearing the first-generation Oculus Rift headset.

Suppose the screen in front of the user’s eyes shows a fixed image in the headset. If the user rotates his head, then the image will be perceived as being attached to the head. This would occur, for example, if you rotate your head while using the Viewmaster (recall Figure 1.25(b)). If you would like to instead perceive the image as part of a fixed world around you, then the image inside the headset must change to compensate as you rotate your head. The surrounding virtual world should be counter-rotated, the meaning of which should become more precise after reading Section 3.4. Once we agree that such transformations are necessary, it becomes a significant engineering challenge to estimate the amount of head and eye movement that has occurred and apply the appropriate transformation in a timely and accurate manner. If this is not handled well, then the user could have a poor or unconvincing experience. Worse yet, they could fall prey to VR sickness. This is one of the main reasons why the popularity of VR headsets waned in the 1990s. The component technology was not good enough yet. Fortunately, the situation is much improved at present. For audio, few seemed to bother with this transformation, but for the visual counterpart, it is absolutely critical. One final note is that tracking and applying transformations also becomes necessary in CAVE systems if we want the images on the screens to be altered according to changes in the eye positions inside of the room.

Now that you have a high-level understanding of the common hardware arrangements, we will take a closer look at hardware components that are widely available for constructing VR systems. These are expected to change quickly, with costs decreasing and performance improving. We also expect many new devices to appear in the marketplace in the coming years. In spite of this, the fundamentals in this book remain unchanged. Knowledge of the current technology provides concrete examples to make the fundamental VR concepts clearer.

The hardware components of VR systems are conveniently classified as:

- **Displays (output)**: Devices that stimulate a sense organ.
2.1. HARDWARE

Figure 2.8: Two examples of haptic feedback devices. (a) The Geomagic Phantom allows the user to feel strong resistance when poking into a virtual object with a real stylus. A robot arm provides the appropriate forces. (b) Some game controllers occasionally vibrate.

- **Sensors (input):** Devices that extract information from the real world.
- **Computers:** Devices that process inputs and outputs sequentially.

**Displays**  The purpose of a display is to generate a stimulus for a targeted sense organ. Vision is our dominant sense, and any display constructed for the eye must cause the desired image to be formed on the retina. Because of this importance, Chapters 4 and 5 will explain optical systems and the human vision system, respectively. For CAVE systems, some combination of digital projectors and mirrors are used. Due to the plummeting costs, an array of large-panel displays may alternatively be employed. For headsets, a smartphone display can be placed close to the eyes and brought into focus using one magnifying lens for each eye. Screen manufacturers are currently making custom displays for VR headsets by leveraging the latest LED display technology from the smartphone industry. Some are targeting one display per eye with frame rates above 90Hz and over two megapixels per eye. Reasons for this are explained in Chapter 5. More exotic displays, which are primarily in a research-and-development stage include pico projectors, light-field displays, and multi-focal-plane optical systems.

Now imagine displays for other sense organs. Sound is displayed to the ears using classic speaker technology. Bone conduction methods may also be used, which vibrate the skull and propagate the waves to the inner ear; this method appeared recently in Google Glass. Chapter 12 covers the auditory part of VR in detail. For visual and auditory body-mounted displays, the position and orientation of the sense organ must be tracked by sensors to appropriately adapt the stimulus. The orientation part is usually accomplished by an inertial measurement unit or IMU. The main component is a gyroscope, which measures its own rate of rotation; the rate is referred to as angular velocity and has three components. Measurements from the gyroscope are integrated over time to obtain an estimate of the cumulative change in orientation. The resulting error, called drift error, would gradually grow unless other sensors are used. To reduce drift error, IMUs also contain an accelerometer and possibly a magnetometer. Over the years, IMUs have gone from existing only as large mechanical systems in aircraft and missiles to being tiny devices inside of our smartphones; see Figure 2.9. Due to their small size, weight, and cost, IMUs can be easily embedded in wearable devices. They are one of the most important enabling technologies for the current generation of VR headsets and are mainly used for tracking the user’s head orientation.

Digital cameras provide another critical source of information for tracking systems. Like IMUs, they have become increasingly cheap and portable due to the smartphone industry, while at the same time improving in image quality. Cameras enable tracking approaches that exploit line-of-sight visibility. The idea is to place markers on the object to be tracked and find them in the image. Such visibility constraints severely limit the possible object positions and orientations. Standard cameras passively form an image focusing the light through an optical system, much like the human eye. Once the camera calibration parameters are known, an observed marker is known to lie along a ray in space. Cameras are commonly
2.1. HARDWARE

Figure 2.10: (a) The Microsoft Kinect sensor gather both an ordinary RGB image and a depth map (the distance away from the sensor for each pixel). (b) The depth is determined by observing the locations of projected IR dots in an image obtained from an IR camera.

used to track eyes, heads, hands, entire human bodies, and any other objects in the physical world. One of the main difficulties at present is to obtain reliable and accurate performance without placing special markers on the user or objects around the scene.

As opposed to standard cameras, depth cameras work actively by projecting light into the scene and then observing its reflection in the image. This is typically done in the infrared (IR) spectrum so that humans do not notice; see Figure 2.10.

In addition to these sensors, we rely heavily on good-old mechanical switches and potentiometers to create keyboards and game controllers. An optical mouse is also commonly used. One advantage of these familiar devices is that users can rapidly input data or control their characters by leveraging their existing training. A disadvantage is that they might be hard to find or interact with if their faces are covered by a headset.

Computers A computer executes the virtual world generator (VWG). Where should this computer be? Although unimportant for world-fixed displays, the location is crucial for body-fixed displays. If a separate PC is needed to power the system, then fast, reliable communication must be provided between the headset and the PC. This connection is currently made by wires, leading to an awkward tether; current wireless speeds are not sufficient.

To conclude with hardware, Figure 2.12 shows the hardware components for the Oculus Rift DK2, which became available in late 2014. In the lower left corner, you can see a smartphone screen that serves as the display. Above that is a circuit board that contains the IMU, display interface chip, a USB driver chip, a set of chips for driving LEDs on the headset for tracking, and a programmable microcontroller. The lenses, shown in the lower right, are placed so that the smartphone screen appears to be “infinitely far” away, but nevertheless fills most of the field of view of the user. The upper right shows flexible circuits that deliver power to IR LEDs embedded in the headset (they are hidden behind IR-transparent plastic). A camera is used for tracking, and its parts are shown in the center.

2.2 Software

From a developer’s standpoint, it would be ideal to program the VR system by providing high-level descriptions and having the software determine automatically all of the low-level details. In a perfect world, there would be a VR engine, which serves a purpose similar to the game engines available today for creating video games. If the developer follows patterns that many before him have implemented already, then many complicated details can be avoided by simply calling functions from a well-designed software library. However, if the developer wants to try something relatively original, then she would have to design the functions from scratch. This requires a deeper understanding of the VR fundamentals, while also
2.2 SOFTWARE

Unfortunately, we are currently a long way from having fully functional, general-purpose VR engines. As applications of VR broaden, specialized VR engines are also likely to emerge. For example, one might be targeted for immersive cinematography while another is geared toward engineering design. Which components will become more like part of a VR “operating system” and which will become higher level “engine” components? Given the current situation, developers will likely be implementing much of the functionality of their VR systems from scratch. This may involve utilizing a software development kit (SDK) for particular headsets that handles the lowest level operations, such as device drivers, head tracking, and display output. Alternatively, they might find themselves using a game engine that has been recently adapted for VR, even though it was fundamentally designed for video games on a screen. This can avoid substantial effort at first, but then may be cumbersome when someone wants to implement ideas that are not part of standard video games.

What software components are needed to produce a VR experience? Figure 2.13 presents a high-level view that highlights the central role of the Virtual World Generator (VWG). The VWG receives inputs from low-level systems that indicate what the user is doing in the real world. A head tracker provides timely estimates of the user’s head position and orientation. Keyboard, mouse, and game controller events arrive in a queue that are ready to be processed. The key role of the VWG is to maintain enough of an internal “reality” so that renderers can extract the information they need to calculate outputs for their displays.

Virtual world: real vs. synthetic  At one extreme, the virtual world could be completely synthetic. In this case, numerous triangles are defined in a 3D space, along with material properties that indicate how they interact with light, sound, forces, and so on. The field of computer graphics addresses computer-generated images from synthetic models [], and it remains important for VR; see Chapter 7. At the other extreme, the virtual world might be a recorded physical world that was captured using modern cameras, computer vision, and Simultaneous Localization and Mapping (SLAM) techniques; Figure 2.14. Many possibilities exist between the extremes. For example, camera images may be taken of a real object, and then mapped onto a synthetic object in the virtual world. This is called texture mapping, a common operation in computer graphics; see Section 7.3.

Matched motion  The most basic operation of the VWG is to maintain a correspondence between user motions in the real world and the virtual world; see Figure 2.15. In the real world, the user’s motions are confined to a safe region, which we will call the matched zone. Imagine the matched zone as a place where the real and virtual worlds perfectly align. One of the greatest challenges is the mismatch
Software

Figure 2.14: Using both color and depth information from cameras, a 3D model of the world can be extracted automatically using Simultaneous Localization and Mapping (SLAM) techniques. Figure from [22].

of obstacles: What if the user is blocked in the virtual world but not in the real world? The reverse is also possible. In a seated experience, the user sits in a chair while wearing a headset. The matched zone in this case is a small region, such as one cubic meter, in which users can move their heads. Head motions should be matched between the two worlds. If the user is not constrained to a seat, then the matched zone could be an entire room or an outdoor field. Note that safety becomes an issue because the user might spill a drink, hit walls, or fall into pits that exist only in the real world, but are not visible in the virtual world. Larger matched zones tend to lead to greater safety issues. Users must make sure that the matched zone is cleared of dangers in the real world, or the developer should make them visible in the virtual world.

Which motions from the real world should be reflected in the virtual world? This varies among VR experiences. In a VR headset that displays images to the eyes, head motions must be matched so that the visual renderer uses the correct viewpoint in the virtual world. Other parts of the body are less critical, but may become important if the user needs to perform hand-eye coordination or looks at other parts of her body and expects them to move naturally.

Locomotion In many VR experiences, users want to move well outside of the matched zone. This motivates locomotion, which means moving oneself in the virtual world, while this motion is not matched in the real world. Imagine you want to explore a virtual city while remaining seated in the real world. How should this be achieved? You could pull up a map and point to where you want to go, with a quick teleportation operation sending you to the destination. A popular option is to move oneself in the virtual world by operating a game controller, mouse, or keyboard. By pressing buttons or moving knobs, your self in the virtual world could be walking, running, jumping, swimming, flying, and so on. You could also climb aboard a vehicle in the virtual world and operate its controls to move yourself. These operations are certainly convenient, but often lead to sickness because of a mismatch between your vestibular and visual senses. See Sections 2.3 and 10.1.

Physics The VWG handles the geometric aspects of motion by applying the appropriate mathematical transformations. In addition, the VWG usually implements some physics so that as time progresses, the virtual world behaves like the real world. In most cases, the basic laws of mechanics should govern how objects move in the virtual world. For example, if you drop an object, it should accelerate to the ground due to gravitational force acting on it. One important component is a collision detection algorithm, which determines whether two or more bodies are intersecting in the virtual world. If a new collision occurs, then an appropriate response is needed. For example, suppose the user pokes his head through a wall in the virtual world. Should the head in the virtual world be stopped, even though it continues to move in the real world? To it more complex, what should happen if you unload a dump truck full of basketballs into a busy street in the virtual world?
Simulated physics can become quite challenging, and is a discipline in itself. There is no limit to the complexity.

In addition to handling the motions of moving objects, the physics must also take into account how potential stimuli for the displays are created and propagate through the virtual world. How does light propagate through the environment? How does light interact with the surfaces in the virtual world? What are the sources of light? How do sound and smells propagate? These correspond to rendering problems, which are covered in Chapters 7 and 12 for visual and audio cases.

**Networked experiences** In the case of a networked VR experience, a shared virtual world is maintained by a server. Each user has a distinct matched zone. Their matched zones might overlap in a real world, but one must then be careful so that they avoid unwanted collisions. Most often, these zones are disjoint and distributed around the Earth. Within the virtual world, user interactions, including collisions, must be managed by the VWG. If multiple users are interacting in a social setting, then the burdens of matched motions may increase. As users see each other, they could expect to see eye motions, facial expressions, and body language; see Section 10.4.

**Developer choices for VWGs** To summarize, a developer could start with a basic Software Development Kit (SDK) from a VR headset vendor and then build her own VWG from scratch. The SDK should provide the basic drivers and an interface to access tracking data and make calls to the graphical rendering libraries. In this case, the developer must build the physics of the virtual world from scratch, handling problems such as avatar movement, collision detection, lighting models, and audio. This gives the developer the greatest amount of control and ability to optimize performance; however, it may come in exchange for a difficult implementation burden. In some special cases, it might not be too difficult. For example, in the case of the Google Street viewer (recall Figure 1.8), the “physics” is simple: The viewing location needs to jump between panoramic images in a comfortable way while maintaining a sense of location on the Earth. In the case of telepresence using a robot, the VWG would have to take into account movements in the physical world. Failure to handle collision detection could result in a broken robot (or human!).

At the other extreme, a developer may use a ready-made VWG that is customized to make a particular VR experience by choosing menu options and writing high-level scripts. Examples available today are OpenSimulator, Vizard by World-Viz, Unity 3D, and Unreal Engine by Epic Games. The latter two are game engines that were adapted to work for VR, and are by far the most popular among current VR developers. The first one, OpenSimulator, was designed as an open-source alternative to Second Life for building a virtual society of avatars. Using such higher-level engines make it easy for developers to make a VR experience in little time; however, the drawback is that it is harder to make unusual experiences that were not imagined by the engine builders.

### 2.3 Human Physiology and Perception

Our bodies were not designed for VR. By applying artificial stimulation to the senses, we are disrupting the operation of biological mechanisms that have taken hundreds of millions of years to evolve in a natural environment. We are also providing input to the brain that is not exactly consistent with all of our other life experiences. In some instances, our bodies may adapt to the new stimuli. This could cause us to become unaware of flaws in the VR system. In other cases, we might develop heightened awareness or the ability to interpret 3D scenes that were once difficult or ambiguous. Unfortunately, there are also many cases where our bodies react by increased fatigue or headaches, partly because the brain is working harder than usual to interpret the stimuli. Finally, the worst case is the onset of VR sickness, which typically involves symptoms of dizziness and nausea.

Perceptual psychology is the science of understanding how the brain converts sensory stimulation into perceived phenomena. Here are some typical questions that arise in VR and fall under this umbrella:

- How far away does that object appear to be?
- How much video resolution is needed to avoid seeing pixels?
- How many frames per second are enough to perceive motion as continuous?
- Is the user’s head appearing at the proper height in the virtual world?
- Where is that virtual sound coming from?
- Why I am feeling nauseated?
- Why is one experience more tiring than another?
- What is presence?

To answer these questions and more, we must understand several things: 1) basic physiology of the human body, including sense organs and neural pathways, 2) the key theories and insights of experimental perceptual psychology, and 3) the interference of the engineered VR system with our common perceptual processes and the resulting implications or side-effects.

The perceptual side of VR often attracts far too little attention among developers. In the real world, perceptual processes are mostly invisible to us. Think about how much effort it requires to recognize a family member. When you see someone you know well, the process starts automatically, finishes immediately, and seems to require no effort. Scientists have conducted experiments that reveal how much work actually occurs in this and other perceptual processes. Through brain lesion studies, they are able to see the effects when a small part of the brain
is not functioning correctly. Some people suffer from prosopagnosia, which makes them unable to recognize the faces of familiar people, including themselves in a mirror, even though nearly everything else functions normally. Scientists are also able to perform single-unit recordings, mostly on animals, which reveal the firings of a single neuron in response to sensory stimuli. Imagine, for example, a single neuron that fires whenever you see a sphere.

Optical illusions One of the most popular ways to appreciate the complexity of our perceptual processing is to view optical illusions. These yield surprising results and are completely unobtrusive. Each one is designed to reveal some shortcoming of our visual system by providing a stimulus that is not quite consistent with ordinary stimuli in our everyday lives. Figure 2.16 shows two. These should motivate you to appreciate the amount of work that our sense organs and neural structures are doing to fill in missing details and make interpretations based on the context of our life experiences and existing biological structures. Interfering with these without understanding them is not wise!

Classification of senses Perception and illusions are not limited to our eyes. Figure 2.17 shows a classification of our basic senses. Recall that a sensor converts an energy source into signals in a circuit. In the case of our bodies, this means that a stimulus is converted into neural impulses. For each sense, Figure 2.17 indicates the type of energy for the stimulus and the receptor that converts the stimulus into neural impulses. Think of each receptor as a sensor that targets a particular kind of stimulus. This is referred to as sensory system selectivity. In each eye, over 100 million photoreceptors target electromagnetic energy precisely in the frequency range of visible light. Different kinds even target various colors and light levels; see Section 5.1. The auditory, touch, and balance senses involve motion, vibration, or gravitational force; these are sensed by mechanoreceptors. The sense of touch additionally involves thermoreceptors to detect change in temperature. Our balance sense helps us to know which way our head is oriented, including sensing the direction of “up”. Finally, our sense of taste and smell is grouped into one category that relies on chemoreceptors, which provide signals based on chemical composition of matter appearing on our tongue or in our nasal passages.

Note that senses have engineering equivalents, most of which appear in VR systems. Imagine you are designing a humanoid telepresence robot, which you expect to interface with through a VR headset. You could then experience life through your surrogate robotic self. Digital cameras would serve as its eyes, and microphones would be the ears. Pressure sensors and thermometers could be installed to give a sense of touch. For balance, we can install an IMU. In fact, the human vestibular organs and modern IMUs bear a striking resemblance in terms of the signals they produce; see Section 8.4. We could even install chemical sensors, such as a pH meter, to measure aspects of chemical composition to provide taste and smell.

Big brains Perception happens after the sense organs convert the stimuli into neural impulses. According to latest estimates [3], human bodies contain around 86 billion neurons. Around 20 billion are devoted to the part of the brain called the cerebral cortex, which handles perception and many other high-level functions such as attention, memory, language, and consciousness. It is a large sheet of neurons around three millimeters thick and is heavily folded so that it fits into our skulls. In case you are wondering where we lie among other animals, a roundworm, fruit fly, and rat have 302, 100 thousand, and 200 million neurons, respectively. An elephant has over 250 billion neurons, which is more than us! Only mammals have a cerebral cortex. The cerebral cortex of a rat has around 20 million neurons. Cats and dogs are at 160 and 300 million, respectively. A gorilla has around 4 billion. A type of dolphin called the long-finned pilot whale has an estimated 37 billion neurons in its cerebral cortex, making it roughly twice as large as the human...
2.3. HUMAN PHYSIOLOGY AND PERCEPTION

Once in the cerebral cortex, the signals from sensors are combined with anything else from our life experiences that may become relevant for making an interpretation of the stimuli. Various perceptual phenomena occur, such as recognizing a face or identifying a song. Information or concepts that appear in the cerebral cortex tend to represent a global picture of the world around us. Surprisingly, topographic mapping methods reveal that spatial relationships among receptors are maintained in some cases among the distribution of neurons. Also, recall from Section 1.1 that place cells and grid cells encode spatial maps of familiar environments.

**Proprioception** In addition to information from senses and memory, we also use proprioception, which is the ability to sense the relative positions of parts of our bodies and the amount of muscular effort being involved in moving them. Close your eyes and move your arms around in an open area. You should have an idea of where your arms are located, although you might not be able to precisely reach out and touch your fingertips together without using your eyes. This information is so important to our brains that the motor cortex, which controls body motion, sends signals called *efference copies* to other parts of the brain to communicate what motions have been executed. Proprioception is effectively another kind of sense. Continuing our comparison with robots, it corresponds to having *encoders* on joints or wheels, to indicate how far they have moved. One interesting implication of proprioception is that you cannot tickle yourself because you know where your fingers are moving; however, if someone else tickles you, then you do not have access to their efference copies. The lack of this information is crucial to the tickling sensation.

**Fusion of senses** Signals from multiple senses and proprioception are being processed and combined with our experiences by our neural structures throughout our lives. In ordinary life, without VR or drugs, our brains interpret these combinations of inputs in coherent, consistent, and familiar ways. Any attempt to interfere with these operations is likely to cause a mismatch among the data from our senses. The brain may react in a variety of ways. It could be the case they we are not consciously aware of the conflict, but we may become fatigued or develop a headache. Even worse, we could develop symptoms of dizziness or nausea. In other cases, the brain might react by making us so consciously aware of the conflict that we immediately understand that the experience is artificial. This would correspond to a case in which the VR experience is failing to convince someone that they are present in a virtual world. To make an effective and comfortable VR experience, trials with human subjects are essential to understand how the brain reacts. It is practically impossible to predict what would happen in an unknown scenario, unless it is almost identical to other well-studied scenarios.

One of the most important examples of bad sensory conflict in the context of VR is *vection*, which is the illusion of self motion. The conflict arises when your vision sense reports to your brain that you are accelerating, but your balance sense...
reports that you are motionless. As you walk down the street, the balance and vision senses are in harmony. You might have experienced vection before, even without VR. If you are stuck in traffic or stopped at a train station, you might have felt as if you are moving backwards while seeing a vehicle in your periphery that is moving forward. In the 1890s, Amariah Lake constructed an amusement park ride that consisted of a swing that remains at rest while the entire room surrounding the swing rocks back-and-forth (Figure 2.20). In VR, vection is caused by the locomotion operation described in Section 2.2. For example, if you accelerate yourself forward using a controller, rather than moving forward in the real world, then you perceive acceleration with your eyes, but not your vestibular organ. For strategies to alleviate this problem, see Section 10.1.

Adaptation A universal feature of our sensory systems is adaptation, which means that the perceived effect of stimuli changes over time. This may happen with any of our senses and over a wide spectrum of time intervals. For example, the perceived loudness of motor noise in an aircraft or car decreases within minutes. In the case of vision, the optical system of our eyes and the photoreceptor sensitivities adapt to change perceived brightness. Over long periods of time, perceptual training can lead to adaptation. In military training simulations, sickness experienced by soldiers is greatly reduced by regular exposure[]. Anecdotally, the same seems to be true of experienced video game players. Those who have spent many hours and days in front of large screens playing first-person shooter games apparently experience less vection when locomoting themselves in VR.

Adaptation therefore becomes a crucial factor for VR. Through repeated exposure, developers may become comfortable with an experience that is nauseating to a newcomer. This gives them a terrible bias while developing an experience; recall from Section 1.1 the problem of confusing the scientist with the lab subject in the VR experiment. On the other hand, through repeated, targeted training they may be able to improve their debugging skills by noticing flaws in the system that an “untrained eye” would easily miss. Common examples include:

- A large amount of tracking latency has appeared, which interferes with the perception of stationarity.
- The left and right eye views are swapped.
- Objects appear to one eye but not the other.
- One eye view has significantly more latency than the other.
- Straight lines are slightly curved due to uncorrected warping in the optical system.

This disconnect between the actual stimulus and your perception of the stimulus leads to the next topic.
2.3. HUMAN PHYSIOLOGY AND PERCEPTION

Psychophysics

Psychophysics is the scientific study of perceptual phenomena that are produced by physical stimuli. For example, under what conditions would someone call an object “red”? The stimulus corresponds to light entering the eye, and the perceptual phenomenon is the concept of “red” forming in the brain. Other examples of perceptual phenomena are “straight”, “larger”, “louder”, “tickles”, and “sour”. Figure 2.21 shows a typical scenario in a psychophysical experiment. As one parameter is varied, such as the frequency of a light, there is usually a range of values for which subjects cannot reliably classify the phenomenon. For example, there may be a region where they are not sure whether the light is red. At one extreme, they may consistently classify it as “red” and at the other extreme, they consistently classify it as “not red”. For the region in between, the probability of detection is recorded, which corresponds to the frequency with which it is classified as “red”.

Stevens’ power law

One of the most known results from psychophysics is Stevens’s power law, which characterizes the relationship between the magnitude of a physical stimulus and its perceived magnitude. The hypothesis is that an exponential relationship occurs over a wide range of sensory systems and stimuli:

\[ p = cm^x \]  \hspace{1cm} (2.1)

in which

- \( m \) is the magnitude or intensity of the stimulus,
- \( p \) is the perceived magnitude,
- \( x \) relates the actual magnitude to the perceived magnitude, and is the most important part of the equation, and
- \( c \) is an uninteresting constant that depends on units.

Note that for \( x = 1 \), (2.1) is a linear relationship, \( p = cm \); see Figure 2.22. An example of this is our perception of the length of an isolated line segment directly in front of our eyes. The length we perceive is proportional to its actual length. The more interesting cases are when \( x \neq 1 \). For the case of perceiving the brightness of a target in the dark, \( x = 0.33 \), which implies that a large increase in brightness is perceived as a smaller increase. In the other direction, our perception of electric shock as current through the fingers yields \( x = 3.5 \). A little more shock is a lot more uncomfortable!

Just noticeable difference

Another key psychophysical concept is the Just Noticeable Difference (JND). This is the amount that the stimulus needs to be changed so that subjects would perceive it to have changed in at least 50 percent of trials. For a large change, all or nearly all subjects would report a change. If the change is too small, then none or nearly none of the subjects would notice. The experimental challenge is to vary the amount of change until the chance of someone reporting a change is 50 percent.

Consider the JND for a stimulus with varying magnitude, such as brightness. How does the JND itself vary as the magnitude varies? This relationship is cap-
tured by Weber’s law:

\[
\frac{\Delta m}{m} = c,
\]

(2.2)
in which \(\Delta m\) is the JND, \(m\) is the magnitude stimulus, and \(c\) is a constant.

**Design of experiments** VR disrupts the ordinary perceptual processes of its users. It should be clear from this section that proposed VR systems and experi-

ences need to be evaluated on users to understand whether they are yielding the desired effect while also avoiding unwanted side effects. This amounts to applying

the scientific method to make observations, formulate hypotheses, and design ex-

periments that determine their validity. When human subjects are involved, this

becomes extremely challenging. How many subjects are enough? What happens

if they adapt to the experiment? How does their prior world experience affect

the experiment? What if they are slightly sick the day that they try the experi-

ment? What did they eat for breakfast? The answers to these questions could
dramatically affect the outcome.

It gets worse. Suppose they already know your hypothesis going into the

experiment. This will most likely bias their responses. Also, what will the data

from the experiment look like? Will you ask them to fill out a questionnaire, or

will you make inferences about their experience from measured data such as head

motions, heart rate, and skin conductance? These choices are also critical. See

Chapter 11 for more on this topic.

**Further Reading**

VR sickness survey paper: [24]

Dynamical simulation literature:

More neuroscience: [41]
Chapter 3
The Geometry of Virtual Worlds

Section 2.2 introduced the Virtual World Generator (VWG), which maintains the geometry and physics of the virtual world. This chapter covers the geometry part, which is needed to make models and move them around. The models could include the walls of a building, furniture, clouds in the sky, the user’s avatar, and so on. Section 3.1 covers the basics of how to define consistent, useful models. Section 3.2 explains how to apply mathematical transforms that move them around in the virtual world. This involves two components: Translation (changing position) and rotation (changing orientation). Section 3.3 presents the best ways to express and manipulate 3D rotations, which are the most complicated part of moving models. Section 3.4 then covers how the virtual world appears if we try to “look” at it from a particular perspective. This is the geometric component of visual rendering, which is covered in Chapter 7. Finally, Section 3.5 puts all of the transformations together, so that you can see how to go from defining a model to having it appear in the right place on the display.

If you work with high-level engines to build a VR experience, then most of the concepts from this chapter might not seem necessary. You might need only to select options from menus and write simple scripts. However, an understanding of the basic transformations, such as how to express 3D rotations or move a camera viewpoint, is essential to making the software do what you want. Furthermore, if you want to build virtual worlds from scratch, or at least want to understand what is going on under the hood of a software engine, then this chapter is critical.

Figure 3.1: Points in the virtual world are given coordinates in a right-handed coordinate system in which the y axis is pointing upward. The origin (0, 0, 0) lies at the point where axes intersect. Also shown is a 3D triangle defined by its three vertices, each of which is a point in $\mathbb{R}^3$.

3.1 Geometric Models

We first need a virtual world to contain the geometric models. For our purposes, it is enough to have a 3D Euclidean space with Cartesian coordinates. Therefore, let $\mathbb{R}^3$ denote the virtual world, in which every point is represented as a triple of real-valued coordinates: $(x, y, z)$. The coordinate axes of our virtual world are shown in Figure 3.1. We will consistently use right-handed coordinate systems in this book because they represent the predominant choice throughout physics and engineering; however, left-handed systems appear in some places, with the most notable being Microsoft’s DirectX graphical rendering library. In these cases, one of the three axes points in the opposite direction in comparison to its direction in a right-handed system. This inconsistency can lead to hours of madness when writing software; therefore, be aware of the differences and their required conversions if you mix software or models that use both. If possible, avoid mixing right-handed and left-handed systems altogether.

Geometric models are made of surfaces or solid regions in $\mathbb{R}^3$ and contain an infinite number of points. Because representations in a computer must be finite, models are defined in terms of primitives in which each represents an infinite set of points. The simplest and most useful primitive is a 3D triangle, as shown in Figure 3.1. A planar surface patch that corresponds to all points “inside” and on the boundary of the triangle is fully specified by the coordinates of the triangle vertices:

$$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).$$

To model a complicated object or body in the virtual world, numerous triangles can be arranged into a mesh, as shown in Figure 3.2. This provokes many
3.1. GEOMETRIC MODELS

Figure 3.2: A geometric model of a dolphin, formed from a mesh of 3D triangles (from Wikipedia user Chrschn).

important questions:

1. How do we specify how each triangle “looks” whenever viewed by a user in VR?
2. How do we make the object “move”?
3. If the object surface is sharply curved, then should we use curved primitives, rather than trying to approximate the curved object with tiny triangular patches?
4. Is the interior of the object part of the model, or is it represented only by its surface?
5. Is there an efficient algorithm for determining which triangles are adjacent to a given triangle along the surface?
6. Should we avoid duplicating vertex coordinates that are common to many neighboring triangles?

We address these questions in reverse order.

Data structures  Consider listing all of the triangles in a file or memory array. If the triangles form a mesh, then most or all vertices will be shared among multiple triangles. This is clearly a waste of space. Another issue is that we will frequently want to perform operations on the model. For example, after moving an object, can we determine whether it is in collision with another object (see Section 8.2)? A typical low-level task might be to determine which triangles share a common vertex or edge with a given triangle. This might require linearly searching through the triangle list to determine whether they share a vertex or two. If there are millions of triangles, which is not uncommon, then it would cost too much to perform this operation repeatedly.

For these reasons and more, geometric models are usually encoded in clever data structures. The choice of the data structure should depend on which operations will be performed on the model. One of the most useful and common is the doubly connected edge list, also known as half-edge data structure [12, 37]. See Figure 3.3. In this and similar data structures, there are three kinds of data elements: faces, edges, and vertices. These represent two, one, and zero-dimensional parts, respectively, of the model. In our case, every face element represents a triangle. Each edge represents the border of one or two, without duplication. Each vertex is shared between one or more triangles, again without duplication. The data structure contains pointers between adjacent faces, edges, and vertices so that algorithms can quickly traverse the model components in a way that corresponds to how they are connected together.

Inside vs. outside  Now consider the question of whether the object interior is part of the model (recall Figure 3.2). Suppose the mesh triangles fit together perfectly so that every edge borders exactly two triangles and no triangles intersect unless they are adjacent along the surface. In this case, the model forms a complete barrier between the inside and outside of the object. If we were to hypothetically fill the inside with a gas, then it could not leak to the outside. This is an example of a coherent model. Such models are required if the notion of inside or outside is critical to the VWG. For example, a penny could be inside of the dolphin, but not intersecting with any of its boundary triangles. Would this ever need to be detected? If we remove a single triangle, then the hypothetical gas would leak
3.1. GEOMETRIC MODELS

out. There would no longer be a clear distinction between the inside and outside of the object, making it difficult to answer the question about the penny and the dolphin. In the extreme case, we could have a single triangle in space. There is clearly no natural inside or outside. Furthermore, the model could be as bad as polygon soup, which is a jumble of triangles that do not fit together nicely and could even have intersecting interiors. In conclusion, be careful when constructing models so that the operations you want to perform later will be logically clear. If you are using a high-level design tool, such as Blender or Maya, to make your models, then coherent models will be automatically built.

Why triangles? Continuing upward through the questions above, triangles are used because they are the simplest for algorithms to handle, especially if implemented in hardware. GPU implementations tend to be biased toward smaller representations so that a compact list of instructions can be applied to numerous model parts in parallel. It is certainly possible to use more complicated primitives, such as quadrilaterals, splines, and semi-algebraic surfaces [15, 21, 35]. This could lead to smaller model sizes, but often comes at the expense of greater computational cost for handling each primitive. For example, it is much harder to determine whether two spline surfaces are colliding, in comparison to two 3D triangles.

Stationary vs. movable models There will be two kinds of models in the virtual world $\mathbb{R}^3$:

- **Stationary models**, which keep the same coordinates forever. Typical examples are streets, floors, and buildings.
- **Movable models**, which can be transformed into various positions and orientations. Examples include vehicles, avatars, and small furniture.

Motion can be caused in a number of ways. Using a tracking system (Chapter 9), the model might move to match the user’s motions. Alternatively, the user might operate a controller to move objects in the virtual world, including a representation of himself. Finally, objects might move on their own according to the laws of physics in the virtual world. Section 3.2 will cover the mathematical operations that move models to the their desired places, and Chapter 8 will describe velocities, accelerations, and other physical aspects of motion.

Choosing coordinate axes One often neglected point is the choice of coordinates for the models, in terms of their placement and scale. If these are defined cleverly at the outset, then many tedious complications can be avoided. If the virtual world is supposed to correspond to familiar environments from the real world, then the axis scaling should match common units. For example, $(1, 0, 0)$ should mean one meter to the right of $(0, 0, 0)$. It is also wise to put the origin $(0, 0, 0)$ in a convenient location. Commonly, $y = 0$ corresponds to the floor of a building or sea level of a terrain. The location of $x = 0$ and $z = 0$ could be in the center of the virtual world so that it nicely divides into quadrants based on sign. Another common choice is to place it in the upper left when viewing the world from above so that all $x$ and $z$ coordinates are nonnegative. For movable models, the location of the origin and the axis directions become extremely important because they affect how the model is rotated. This should become clear in Sections 3.2 and 3.3 as we present rotations.

**Viewing the models** Of course, one of the most important aspects of VR is how the models are going to “look” when viewed on a display. This problem is as we present rotations.

Continuing upward through the questions above, triangles are used because they are the simplest for algorithms to handle, especially if implemented in hardware. GPU implementations tend to be biased toward smaller representations so that a compact list of instructions can be applied to numerous model parts in parallel. It is certainly possible to use more complicated primitives, such as quadrilaterals, splines, and semi-algebraic surfaces [15, 21, 35]. This could lead to smaller model sizes, but often comes at the expense of greater computational cost for handling each primitive. For example, it is much harder to determine whether two spline surfaces are colliding, in comparison to two 3D triangles.

3.2 Changing Position and Orientation

Suppose that a movable model has been defined as a mesh of triangles. To move it, we apply a single transformation to every vertex of every triangle. This section first considers the simple case of translation, followed by the considerably complicated case of rotations. By combining translation and rotation, the model can be placed anywhere, and at any orientation in the virtual world $\mathbb{R}^3$.

**Translations** Consider the following 3D triangle,

$$
(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3),
$$

in which its vertex coordinates are expressed as generic constants.

Let $x_t, y_t,$ and $z_t$ be the amount we would like to change the triangle’s position, along the $x,$ $y,$ and $z$ axes, respectively. The operation of changing position is called translation, and it is given by

$$
(x_1, y_1, z_1) \mapsto (x_1 + x_t, y_1 + y_t, z_1 + z_t)
$$

$$
(x_2, y_2, z_2) \mapsto (x_2 + x_t, y_2 + y_t, z_2 + z_t)
$$

$$
(x_3, y_3, z_3) \mapsto (x_3 + x_t, y_3 + y_t, z_3 + z_t),
$$

in which $a \mapsto b$ denotes that $a$ becomes replaced by $b$ after the transformation is applied. Applying (3.3) to every triangle in a model will translate all of it to the desired location. If the triangles are arranged in a mesh, then it is sufficient to apply the transformation to the vertices alone. All of the triangles will retain their size and shape.
3.2. CHANGING POSITION AND ORIENTATION

Figure 3.4: Every transformation has two possible interpretations, even though the math is the same. Here is a 2D example, in which a triangle is defined in (a). We could translate the triangle by \( x_t = -8 \) and \( y_t = -7 \) to obtain the result in (b). If we instead wanted to hold the triangle fixed but move the origin up by 8 in the \( x \) direction and 7 in the \( y \) direction, then the coordinates of the triangle vertices change the exact same way, as shown in (c).

Relativity Before the transformations become too complicated, we want to caution you about interpreting them correctly. Figures 3.4(a) and 3.4(b) show an example in which a triangle is translated by \( x_t = -8 \) and \( y_t = -7 \). The vertex coordinates are the same in Figures 3.4(b) and 3.4(c). Figure 3.4(b) shows the case we are intended to cover so far: The triangle is interpreted as having moved in the virtual world. However, Figure 3.4(c) shows another possibility: The coordinates of the virtual world have been reassigned so that the triangle is closer to the origin. This is equivalent to having moved the entire world, with the triangle being the only part that does not move. In this case, the translation is applied to the coordinate axes, but they are negated. When we apply more general transformations, this extends so that transforming the coordinate axes results in an inverse of the transformation that would correspondingly move the model. Negation is simply the inverse in the case of translation.

Thus, we have a kind of “relativity”: Did the object move, or did the whole world move around it? This idea will become important in Section 3.4 when we want to change viewpoints. If we were standing at the origin, looking at the triangle, then the result would appear the same in either case; however, if the origin moves, then we would move with it. A deep perceptual problem lies here as well. If we perceive ourselves as having moved, then VR sickness might increase, even though it was the object that moved. In other words, our brains make their best guess as to which type of motion occurred, and sometimes get it wrong.

Getting ready for rotations How do we make the wheels roll on a car? Or turn a table over onto its side? To accomplish these, we need to change the model’s orientation in the virtual world. The operation that changes the orientation is called rotation. Unfortunately, rotations in three dimensions are much more complicated than translations, leading to countless frustrations for engineers and developers. To improve the clarity of 3D rotation concepts, we first start with a simpler problem: 2D linear transformations.

Consider a 2D virtual world, in which points have coordinates \( (x, y) \). You can imagine this as a vertical plane in our original, 3D virtual world. Now consider a generic two-by-two matrix

\[
M = \begin{bmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{bmatrix}
\]  

(3.4)

in which each of the four entries could be any real number. We will look at what happens when this matrix is multiplied by the point \( (x, y) \), when it is written as a column vector.

Performing the multiplication, we obtain

\[
\begin{bmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix},
\]  

(3.5)

in which \((x', y')\) is the transformed point. Using simple algebra, the matrix multiplication yields

\[
x' = m_{11}x + m_{12}y \\
y' = m_{21}x + m_{22}y,
\]  

(3.6)

Using notation as in (3.3), \( M \) is a transformation for which \((x, y) \mapsto (x', y')\).

Applying the 2D matrix to points Suppose we place two points \((1, 0)\) and \((0, 1)\) in the plane. They lie on the \( x \) and \( y \) axes, respectively, at one unit of distance from the origin \((0, 0)\). Using vector spaces, these two points would be the standard unit basis vectors (sometimes written as \( i \) and \( j \)). Watch what happens if we substitute them into (3.5):

\[
\begin{bmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} m_{11} \\ m_{21}\end{bmatrix}
\]  

(3.7)

and

\[
\begin{bmatrix}
m_{11} & m_{12} \\
m_{21} & m_{22}
\end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} m_{12} \\ m_{22}\end{bmatrix}.
\]  

(3.8)

These special points simply select the column vectors on \( M \). What does this mean? If \( M \) is applied to transform a model, then each column of \( M \) indicates precisely how each coordinate axis is changed.

Figure 3.5 illustrates the effect of applying various matrices \( M \) to a model. Starting with the upper right, the identity matrix does not cause the coordinates to change: \((x, y) \mapsto (x, y)\). The second example causes a flip as if a mirror were placed at the \( y \) axis. In this case, \((x, y) \mapsto (-x, y)\). The second row shows examples of scaling. The matrix on the left produces \((x, y) \mapsto (2x, 2y)\), which doubles the size. The matrix on the right only stretches the model in the \( y \) direction, causing an aspect ratio distortion. In the third row, it might seem that...
3.2. CHANGING POSITION AND ORIENTATION

The matrix on the left produces a mirror image with respect to both \( x \) and \( y \) axes. This is true, except that the mirror image of a mirror image restores the original. Thus, this corresponds to the case of a 180-degree (\( \pi \) radians) rotation, rather than a mirror image. The matrix on the right produces a shear along the \( x \) direction: \( (x, y) \rightarrow (x + y, y) \). The amount of displacement is proportional to \( y \). In the bottom row, the matrix on the left shows a skew in the \( y \) direction. The final matrix might at first appear to cause more skewing, but it is degenerate.

**Only some matrices produce rotations** The examples in Figure 3.5 span the main qualitative differences between various two-by-two matrices \( M \). Two of them were rotation matrices: the identity matrix, which is 0 degrees of rotation, and the 180-degree rotation matrix. Among the set of all possible \( M \), which ones are valid rotations? We must ensure that the model does not become distorted. This is achieved by ensuring that \( M \) satisfies the following rules:

1. No stretching of axes.
2. No shearing.
3. No mirror images.

If none of these rules is violated, then the result is a rotation.

To satisfy the first rule, the columns of \( M \) must have unit length:

\[
\begin{align*}
m_{11}^2 + m_{21}^2 &= 1 \\
m_{12}^2 + m_{22}^2 &= 1
\end{align*}
\]  
(3.9)

The scaling and shearing transformations in Figure 3.5 violated this. To satisfy the second rule, the coordinate axes must remain perpendicular. Otherwise, shearing occurs. Since the columns of \( M \) indicate how axes are transformed, the rule implies that their inner (dot) product is zero:

\[
m_{11}m_{12} + m_{21}m_{22} = 0.
\]  
(3.10)

The shearing transformations in Figure 3.5 violate this rule, which clearly causes right angles in the model to be destroyed.

Satisfying the third rule requires that the determinant of \( M \) is positive. After satisfying the first two rules, the only possible remaining determinants are 1 (the normal case) and −1 (the mirror-image case). Thus, the rule implies that:

\[
\det \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = m_{11}m_{22} - m_{12}m_{21} = 1.
\]  
(3.11)

The mirror image example in Figure 3.5 results in \( \det M = -1 \).
3.2. CHANGING POSITION AND ORIENTATION

Now we try to describe the set of all 3D rotations by following the same general template as the 2D case. The matrix from (3.4) is extended from 2D to 3D, resulting in 9 components:

\[ M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}. \]  

Thus, we start with 9 DOFs and want to determine what matrices remain as valid rotations. Follow the same three rules from the 2D case. The columns must have unit length. For example, \( m_{11}^2 + m_{21}^2 + m_{31}^2 = 1 \). This means that the components of each column must lie on a unit sphere. Thus, the unit-length rule reduces the DOFs from 9 to 6. By following the second rule to ensure perpendicular axes result, the pairwise inner products of the columns must be zero. For example, by choosing the first two columns, the constraint is

\[ m_{11} m_{13} + m_{21} m_{23} + m_{31} m_{33} = 0. \]  

We must also apply the rule to the remaining pairs: The second and third columns, and then the first and third columns. Each of these cases eliminates a DOF, resulting in only 3 DOFs remaining. To avoid mirror images, the constraint \( \det M = 1 \) is applied, which does not reduce the DOFs.

Finally, we arrive at a set of matrices that must satisfy the algebraic constraints; however, they unfortunately do not fall onto a nice circle or sphere. We only know that there are 3 degrees of rotational freedom, which implies that it should be possible to pick three independent parameters for a 3D rotation, and then derive all 9 elements of (3.14) from them.

**Yaw, pitch, and roll** One of the simplest ways to parameterize 3D rotations is to construct them from “2D-like” transformations, as shown in Figure 3.7. First consider a rotation about the z-axis. Let roll be a counterclockwise rotation of \( \gamma \)
about the $z$-axis. The rotation matrix is given by
\[
R_z(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\] (3.16)

The upper left of the matrix looks exactly like the 2D rotation matrix (3.13), except that $\theta$ is replaced by $\gamma$. This causes yaw to behave exactly like 2D rotation in the $xy$ plane. The remainder of $R(\gamma)$ looks like the identity matrix, which causes $z$ to remain unchanged after a roll.

Similarly, let pitch be a counterclockwise rotation of $\beta$ about the $x$-axis:
\[
R_x(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta \\ 0 & \sin \beta & \cos \beta \end{bmatrix}.
\] (3.17)

In this case, points are rotated with respect to $y$ and $z$ while the $x$ coordinate is left unchanged.

Finally, let yaw be a counterclockwise rotation of $\alpha$ about the $y$-axis:
\[
R_y(\alpha) = \begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix}.
\] (3.18)

In this case, rotation occurs with respect to $x$ and $z$ while leaving $y$ unchanged.

Combining rotations Each of (3.16), (3.17), and (3.18) provides a single DOF of rotations. The yaw, pitch, and roll rotations can be combined sequentially to attain any possible 3D rotation:
\[
R(\alpha, \beta, \gamma) = R_y(\alpha)R_x(\beta)R_z(\gamma).
\] (3.19)

In this case, the range of $\alpha$ is from 0 to $2\pi$; however, the pitch $\beta$ need only range from $-\pi/2$ to $\pi/2$ while nevertheless reaching all possible 3D rotations.

Be extra careful when combining rotations in a sequence because the operations are not commutative. For example, a yaw by $\pi/2$ followed by a pitch by $\pi/2$ does not produce the same result as the pitch followed by the yaw. You can easily check this by substituting $\pi/2$ into (3.17) and (3.18), and observing how the result depends on the order of matrix multiplication. The 2D case is commutative because the rotation axis is always the same, allowing the rotation angles to additively combine. Having the wrong matrix ordering is one of the most frustrating problems when writing software for VR.

Matrix multiplications are “backwards” Which operation is getting applied to the model first when we apply a product of matrices? Consider rotating a point $p = (x, y, z)$. We have two rotation matrices $R$ and $Q$. If we rotate $p$ using $Q$, we obtain $p' = Rp$. If we then apply $Q$, we get $p'' = Qp'$. Now suppose that we instead want to first combine the two rotations and then apply them to $p$ to get $p''$. Programmers are often tempted to combine them as $RQ$ because we read from left to right and also write sequences in this way. However, it is backwards for linear algebra because $Rp$ is already acting from the left side. Thus, it “reads” from right to left. We therefore must combine the rotations as $QR$ to obtain $p'' = QRp$. Later in this chapter, we will be chaining together several matrix transforms. Read them from right to left to understand what they are doing!

Translation and rotation in one matrix It would be convenient to apply both translation and rotation together in a single operation. Suppose we want to apply a rotation matrix $R$, and follow it with a translation by $(x_t, y_t, z_t)$. Algebraically, this is
\[
\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix}
\] (3.20)

Although there is no way to form a single 3 by 3 matrix to accomplish both operations, it can be done by increasing the matrix dimensions by one. Consider the following 4 by 4 homogeneous transform matrix:
\[
T_{rb} = \begin{bmatrix} R & x_t \\ 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\] (3.21)

The notation $T_{rb}$ is used to denote that the matrix is a rigid body transform, meaning that it does not distort objects. A homogeneous transform matrix could include other kinds of transforms, which will appear in Section 3.5.

The same result as in (3.20) can be obtained by performing multiplication with (3.21) as follows:
\[
\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \begin{bmatrix} R & x_t \\ 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}
\] (3.22)

Because of the extra dimension, we extended the point $(x, y, z)$ by one dimension, to obtain $(x, y, z, 1)$. Note that (3.21) represents rotation followed by translation, not the other way around. Translate and rotation do not commute; therefore, this is an important point.

Inverting transforms We frequently want to invert (or undo) transformations. For a translation $(x_t, y_t, z_t)$, we simply apply the negation $(-x_t, -y_t, -z_t)$. For a

---

1 Perhaps coders who speak Arabic or Hebrew are not confused about this.
3.2. CHANGING POSITION AND ORIENTATION

The result after rotation by \( \pi/4 \) (45 degrees), following by translation by \( x_t = 2 \). (c) The result after reversing the order: Translation by \( x_t = 2 \), following by rotation by \( \pi/4 \).

Figure 3.8: (a) A rigid model that is contained in a one-by-one square. (b) The result after rotation by \( \pi/4 \) (45 degrees), following by translation by \( x_t = 2 \). (c) The result after reversing the order: Translation by \( x_t = 2 \), following by rotation by \( \pi/4 \).

general matrix transform \( M \), we apply the matrix inverse \( M^{-1} \) (if it exists). This is often complicated to calculate. Fortunately, inverses are much simpler for our cases of interest. In the case of a rotation matrix \( R \), the inverse is equal to the transpose \( R^{-1} = R^T \). To invert the homogeneous transform matrix (3.21), it is tempting to write

\[
\begin{bmatrix}
-x_t \\
-y_t \\
-\z_t \\
0 \\
0 \\
1
\end{bmatrix}.
\]

This will undo both the translation and the rotation; however, the order is wrong. Remember that these operations are not commutative, which implies that order must be correctly handled. See Figure 3.8. The algebra for very general matrices (part of noncommutative group theory) works out so that the inverse of a product of matrices reverses their order:

\[
(ABC)^{-1} = C^{-1}B^{-1}A^{-1}.
\]

This can be seen by putting the inverse next to the original product: \( ABCC^{-1}B^{-1}A^{-1} \). In this way, \( C \) cancels with its inverse, followed by \( B \) and its inverse, and finally \( A \) and its inverse. If the order were wrong, then these cancellations would not occur.

The matrix \( T_{rb} \) (from 3.21) applies the rotation first, followed by translation. Applying (3.23) undoes the rotation first and then translation, without reversing the order. Thus, the inverse of \( T_{rb} \) is

\[
\begin{bmatrix}
0 & 0 & \frac{-x_t}{R^T} \\
0 & 1 & \frac{-y_t}{R^T} \\
0 & 0 & \frac{-z_t}{R^T}
\end{bmatrix}.
\]

The matrix on the left undoes the translation (with no rotation). After that, the matrix on the left undoes the rotation (with no translation).

3.3 Axis-Angle Representations of Rotation

As observed in Section 3.2, 3D rotation is complicated for several reasons: 1) Nine matrix entries are specified in terms of only three independent parameters, and with no simple parameterization, 2) the axis of rotation is not the same every time, and 3) the operations are noncommutative, implying that the order of matrices is crucial. None of these problems existed for the 2D case.

Kinematic singularities An even worse problem arises when using yaw, pitch, roll angles (and related Euler-angle variants). Even though they start off being intuitively pleasing, the representation becomes degenerate, leading to kinematic singularities that are nearly impossible to visualize. An example will be presented shortly. To prepare for this, recall how we represent locations on the Earth. These are points in \( \mathbb{R}^3 \), but are represented with longitude and latitude coordinates. Just like the limits of yaw and pitch, longitude ranges from 0 to 2\( \pi \). (Longitude is usually expressed as 0 to 180 degrees west or east, which is equivalent.) As we travel anywhere on the Earth, the latitude and longitude coordinates behave very much like \( \pm \) \( \alpha \) or \( \pm \gamma \) (which is equivalent). As we travel anywhere on the Earth, the latitude and longitude coordinates behave very much like \( \pm \gamma \) coordinates; however, we tend to stay away from the poles. Near the North Pole, the latitude behaves normally, but the longitude could vary a large amount while corresponding to a tiny distance traveled. Recall how a wall map of the world looks near the poles: Greenland is enormous and Antarctica wraps across the entire bottom (assuming it uses a projection that keeps longitude lines straight). The poles themselves are the kinematic singularities: At these special points, you can vary longitude, but the location on the Earth is not changing. One of two DOFs seems to be lost.

The same problem occurs with 3D rotations, but it is harder to visualize due to the extra dimension. If the pitch angle is held at \( \beta = \pi/2 \), then a kind of “North Pole” is reached in which \( \alpha \) and \( \gamma \) vary independently but cause only one DOF (in the case of latitude and longitude, it was one parameter varying but causing DOFs). Here is how it looks when combining the yaw, pitch, and roll matrices:

\[
\begin{bmatrix}
\cos \alpha & 0 & \sin \alpha \\
0 & 1 & 0 \\
-\sin \alpha & 0 & \cos \alpha
\end{bmatrix}
\begin{bmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
\cos(\alpha - \gamma) & \sin(\alpha - \gamma) & 0 \\
-\sin(\alpha - \gamma) & \cos(\alpha - \gamma) & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
3.3. AXI-S-ANGLE REPRESENTATIONS OF ROTATION

Figure 3.9: Euler’s rotation theorem states that every 3D rotation can be considered as a rotation by an angle $\theta$ about an axis through the origin, given by the unit direction vector $v = (v_1, v_2, v_3)$.

The second matrix above corresponds to pitch (3.17) with $\beta = \pi/2$. The result on the right is obtained by performing matrix multiplication and applying a subtraction trigonometric identity. You should observe that the resulting matrix is a function of both $\alpha$ and $\gamma$, but there is one DOF because only the difference $\alpha - \gamma$ affects the resulting rotation. In the video game industry there has been some back-and-forth battles about whether this problem is crucial. In an FPS game, the avatar is usually not allowed to pitch his head all the way to $\pm \pi/2$, thereby avoiding this problem. In VR, it happens all the time that a user could pitch her head straight up or down. The kinematic singularity often causes the virtual world to apparently spin uncontrollably. This phenomenon occurs when sensing and controlling a spacecraft’s orientation using mechanical gimbals; the result is called *gimbal lock*.

The problems can be easily solved with axis-angle representations of rotation. They are harder to learn than yaw, pitch, and roll; however, it is a worthwhile investment because it avoids these problems. Furthermore, many well-written software libraries and game engines work directly with these representations. Thus, to use them effectively, you should understand what they are doing.

The most important insight to solving the kinematic singularity problems is Euler’s rotation theorem (1775), shown in Figure 3.9. Even though the rotation axis may change after rotations are combined, Euler showed that any 3D rotation can be expressed as a rotation $\theta$ about some axis that pokes through the origin. This matches the three DOFs for rotation: It takes two parameters to specify the direction of an axis and one parameter for $\theta$. The only trouble is that conversions back and forth between rotation matrices and the axis-angle representation are somewhat inconvenient. This motivates the introduction of a mathematical object that is close to the axis-angle representation, closely mimics the algebra of 3D rotations, and can even be applied directly to rotate models. The perfect

representation: *Quaternions.*

Two-to-one problem  Before getting to quaternions, it is important point out one annoying problem with Euler’s rotation theorem. As shown in Figure 3.10, it does not claim that the axis-angle representation is unique. In fact, for every 3D rotation other than the identity, there are exactly two representations. This is due to the fact that the axis could “point” in either direction. We could insist that the axis always point in one direction, such as positive $y$, but this does not fully solve the problem because of the boundary cases (horizontal axes). Quaternions, which are coming next, nicely handle all problems with 3D rotations except this one, which is unavoidable.

Quaternions were introduced in 1843 by William Rowan Hamilton. When seeing them the first time, most people have difficulty understanding their peculiar algebra. Therefore, we will instead focus on precisely which quaternions correspond to which rotations. After that, we will introduce some limited quaternion algebra. The algebra is much less important for developing VR systems, unless you want to implement your own 3D rotation library. The correspondence between quaternions and 3D rotations, however, is crucial.

A quaternion $h$ is a 4D vector:

$$q = (a, b, c, d),$$

in which $a$, $b$, $c$, and $d$ can take on real values. Thus, $q$ can be considered as a point in $\mathbb{R}^4$. It turns out that we will only use *unit quaternions*, which means that

$$a^2 + b^2 + c^2 + d^2 = 1$$

must always hold. This should remind you of the equation of a unit sphere ($x^2 + y^2 + z^2 = 1$), but it is one dimension higher. A sphere is a 2D surface, whereas the set of all unit quaternions is a 3D “hypersurface”, more formally known as a

Figure 3.10: There are two ways to encode the same rotation in terms of axis and angle, using either $v$ or $-v$.
3.3. Axis-Angle Representations of Rotation

<table>
<thead>
<tr>
<th>Quaternion</th>
<th>Axis-Angle</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 0, 0, 0)</td>
<td>(undefined, 0)</td>
<td>Identity rotation</td>
</tr>
<tr>
<td>(0, 1, 0, 0)</td>
<td>(1, 0, 0, π)</td>
<td>Pitch by π</td>
</tr>
<tr>
<td>(0, 0, 1, 0)</td>
<td>(0, 1, 0, π)</td>
<td>Yaw by π</td>
</tr>
<tr>
<td>(0, 0, 0, 1)</td>
<td>(0, 0, 1, π)</td>
<td>Roll by π</td>
</tr>
<tr>
<td>(1/\sqrt{2}, 1/\sqrt{2}, 0)</td>
<td>(1, 0, 0, π/2)</td>
<td>Pitch by π/2</td>
</tr>
<tr>
<td>(1/\sqrt{2}, -1/\sqrt{2}, 0)</td>
<td>(0, 1, 0, π/2)</td>
<td>Yaw by π/2</td>
</tr>
<tr>
<td>(1/\sqrt{2}, -1/\sqrt{2}, 0)</td>
<td>(0, 0, 1, π/2)</td>
<td>Roll by π/2</td>
</tr>
</tbody>
</table>

Figure 3.11: For these cases, you should be able to look at the quaternion and quickly picture the axis and angle of the corresponding 3D rotation.

\[
\begin{align*}
(a, b, c, d) & \quad \text{equivalent} \quad (-a, -b, -c, -d) \\
\text{inverses} & \quad \rightarrow \quad \text{inverses}
\end{align*}
\]

\[
\begin{align*}
(a, b, c, d) & \quad \text{equivalent} \quad (-a, -b, -c, -d) \\
\text{inverses} & \quad \rightarrow \quad \text{inverses}
\end{align*}
\]

Figure 3.12: Simple relationships between equivalent quaternions and their inverses.

\[\text{manifold} \ [5, 25].\] We will use the space of unit quaternions to represent the space of all 3D rotations. Both have 3 DOFs, which seems reasonable.

Let \((v, \theta)\) be an axis-angle representation of a 3D rotation, as depicted in Figure 3.9. Let this be represented by the following quaternion:

\[q = \left(\cos \frac{\theta}{2}, v_1 \sin \frac{\theta}{2}, v_2 \sin \frac{\theta}{2}, v_3 \sin \frac{\theta}{2}\right).\] (3.29)

Think of \(q\) as a data structure that encodes the 3D rotation. It is easy to recover \((v, \theta)\) from \(q\):

\[\theta = 2 \cos^{-1} a \quad \text{and} \quad v = \frac{1}{\sqrt{1 - a^2}}(b, c, d).\] (3.30)

If \(a = 1\), then (3.30) breaks; however, this corresponds to the case of the identity rotation.

You now have the mappings \((v, \theta) \mapsto q\) and \(q \mapsto (v, \theta)\). To test your understanding, Figure 3.11 shows some simple examples, which commonly occur in practice. Furthermore, Figure 3.12 shows some simple relationships between quaternions and their corresponding rotations. The horizontal arrows indicate that \(q\) and \(-q\) represent the same rotation. This is true because of the double representation issue shown in Figure 3.10. Applying (3.29) to both cases establishes their equivalence. The vertical arrows correspond to inverse rotations. These hold because reversing the direction of the axis causes the rotation to be be reversed (rotation by \(\theta\) becomes rotation by \(-\theta\).

How do we apply the quaternion \(h = (a, b, c, d)\) to rotate the model? One way is to use the following conversion into a 3D rotation matrix:

\[
R(h) = \begin{bmatrix}
2(a^2 + b^2) - 1 & 2(bc - ad) & 2(bd + ac) \\
2(bc + ad) & 2(a^2 + c^2) - 1 & 2(cd - ab) \\
2(bd - ac) & 2(cd + ab) & 2(a^2 + d^2) - 1
\end{bmatrix}. \tag{3.31}
\]

A more efficient way exists which avoids converting into a rotation matrix. To accomplish this, we need to define quaternion multiplication. For any two quaternions, \(q_1\) and \(q_2\), let \(q_1 \ast q_2\) denote the product, which is defined as

\[
\begin{align*}
a_3 &= a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\
b_3 &= a_1 b_2 + a_2 b_1 + c_1 d_2 - c_2 d_1 \\
c_3 &= a_1 c_2 + a_2 c_1 + b_1 d_2 - b_2 d_1 \\
d_3 &= a_1 d_2 + a_2 d_1 + b_1 c_2 - b_2 c_1.
\end{align*}
\] (3.32)

In other words, \(q' = q \ast q'\) as defined in (3.32).

Here is a way to rotate the point \((x, y, z)\) using the rotation represented by \(h\). Let \(p = (x, y, z, 1)\), which is done to give the point the same dimensions as a quaternion. Believe it or not, the point is rotated by applying quaternion multiplication as

\[p' = q \ast p \ast q^{-1},\] (3.33)

in which \(q^{-1} = (a, -b, -c, -d)\) (recall from Figure 3.12). The rotated point is \((x', y', z')\), which is taken from the result \(p' = (x', y', z', 1)\).

Here is a simple example for the point \((1, 0, 0, 0)\). Let \(p = (1, 0, 0, 1)\) and consider executing a yaw rotation by \(\pi/2\). According to Figure 3.11, the corresponding quaternion is \(q = (0, 0, 1, 0)\). The inverse is \(q^{-1} = (0, 0, -1, 0)\). After tediously applying (3.32) to calculate (3.33), the result is \(p' = (0, 1, 0, 1)\). Thus, the rotated point is \((0, 1, 0)\), which is a correct yaw by \(\pi/2\).

3.4 Viewing Transformations

This section describes how to transform the models in the virtual world so that they appear on a virtual screen. The main purpose is to set the foundation for graphical rendering, which adds effects due to lighting, material properties, and quantization. Ultimately, the result appears on the physical display. One side effect of these transforms is that they also explain how cameras form images, at least the idealized mathematics of the process. Think of this section as describing a virtual camera that is placed in the virtual world. What should the virtual picture, taken by that camera, look like? To make VR work correctly, the “camera” should actually be one of two virtual human eyes that are placed into the virtual world.
For VR, the position and orientation of eye in the virtual world would it see? Section 3.4 provides transformations that place objects from the virtual world into a virtual screen, based on the particular viewpoint of a virtual eye. A flat rectangular shape is chosen for engineering and historical reasons, even though it does not match the shape of our retinas.

Thus, what should a virtual eye see, based on its position and orientation in the virtual world? Rather than determine precisely what would appear on the retina, which should become clear after Section 4.4, here we merely calculate where the model vertices would appear on a flat, rectangular screen in the virtual world. See Figure 3.13.

An eye’s view Figure 3.14 shows a virtual eye that is looking down the negative z axis. It is placed in this way so that from the eye’s perspective, x increases to the right and y is upward. This corresponds to familiar Cartesian coordinates. The alternatives would be: 1) to face the eye in the positive z direction, which makes the xy coordinates appear backwards, or 2) reverse the z axis, which would unfortunately lead to a left-handed coordinate system. Thus, we have made an odd choice that avoids worse complications.

Suppose that the eye is an object model that we want to place into the virtual world \( \mathbb{R}^3 \) at some position \( e = (e_1, e_2, e_3) \) and orientation given by the matrix

\[
R_{\text{eye}} = \begin{bmatrix}
\hat{x}_1 & \hat{y}_1 & \hat{z}_1 \\
\hat{x}_2 & \hat{y}_2 & \hat{z}_2 \\
\hat{x}_3 & \hat{y}_3 & \hat{z}_3 
\end{bmatrix}.
\]  

(3.34)

If the eyeball in Figure 3.14 were made of triangles, then rotation by \( R_{\text{eye}} \) and translation by \( e \) would be applied to all vertices to place it in \( \mathbb{R}^3 \).

This does not, however, solve the problem of how the virtual world should appear to the eye. Rather than moving the eye in the virtual world, we need to move all of the models in the virtual world to the eye’s frame of reference. This means that we need to apply the inverse transformation. Recall the inverse transform. The inverse rotation is \( R_{\text{eye}}^{-T} \), the transpose of \( R_{\text{eye}} \). The inverse of \( e \) is \( -e \). Applying (3.25) results in the appropriate transform:

\[
T_{\text{eye}} = \begin{bmatrix}
\hat{x}_1 & \hat{x}_2 & \hat{x}_3 & 0 \\
\hat{y}_1 & \hat{y}_2 & \hat{y}_3 & 0 \\
\hat{z}_1 & \hat{z}_2 & \hat{z}_3 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
1 & 0 & 0 & -e_1 \\
0 & 1 & 0 & -e_2 \\
0 & 0 & 1 & -e_3 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]  

(3.35)

Note that \( R_{\text{eye}} \), as shown in (3.34), has been transposed and placed into the left matrix above. Also, the order of translation and rotation have been swapped, which is required for the inverse, as mentioned in Section 3.2.

Following Figure 3.4, there are two possible interpretations of (3.35). As stated, this could correspond to moving all of the virtual world models (corresponding to Figure 3.4(b)). A more appropriate interpretation in the current setting is that the virtual world’s coordinate frame is being moved so that it matches the eye’s frame from Figure 3.14. This corresponds to the case of Figure 3.4(c), which was not the appropriate interpretation in Section 3.2.

Starting from a look-at For VR, the position and orientation of eye in the virtual world are given by a tracking system and possibly controller inputs. By contrast, in computer graphics, it is common to start with a description of where the eye is located and which way it is looking. This is called a look-at, and has the following components:
1. Position of the eye: \( e \)

2. Central looking direction of eye: \( \hat{c} \)

3. Up direction: \( \hat{u} \).

Both \( \hat{c} \) and \( \hat{u} \) are unit vectors. The first direction \( \hat{c} \) corresponds to the center of the view. Whatever \( \hat{c} \) is pointing at should end up in the center of the display. If we want this to be a particular point \( p \) in \( \mathbb{R}^3 \) (see Figure 3.15), then \( \hat{c} \) can be calculated as

\[
\hat{c} = \frac{p - e}{\|p - e\|}.
\]

(3.36)

in which \( \| \cdot \| \) denotes the length of a vector. The result is just the vector from \( e \) to \( p \), but normalized.

The second direction \( \hat{u} \) indicates which way is up. Imagine holding a camera out as if you are about to take a photo and then perform a roll rotation. You can make level ground appear to be slanted or even upside down in the picture. Thus, \( \hat{u} \) indicates the up direction for the virtual camera or eye.

We now construct the resulting transform \( T_{eye} \) from (3.35). The translation components are already determined by \( e \), which was given in the look-at. We need only to determine the rotation \( R_{eye} \), as expressed in (3.34). Recall from Section 3.2 that the matrix columns indicate how the coordinate axes are transformed by the matrix (refer to (3.7) and (3.8)). This simplifies the problem of determining \( R_{eye} \). Each column vector is calculated as

\[
\hat{z} = -\hat{c} \\
\hat{x} = \hat{u} \times \hat{z} \\
\hat{y} = \hat{z} \times \hat{x}.
\]

(3.37)

The minus sign appears for calculating \( \hat{z} \) because the eye is looking down the negative \( z \) axis. The \( \hat{x} \) direction is calculated using the standard cross product \( \hat{z} \).

For the third equation, we could use \( \hat{y} = \hat{u} \); however, \( \hat{z} \times \hat{x} \) will cleverly correct cases in which \( \hat{u} \) generally points upward but is not perpendicular to \( \hat{c} \). The unit vectors from (3.37) are substituted into (3.34) to obtain \( R_{eye} \). Thus, we have all the required information to construct \( T_{eye} \).

**Orthographic projection** Let \( (x, y, z) \) denote the coordinates any point, after \( T_{eye} \) has been applied. What would happen if we took all points and directly projected them into the vertical \( xy \) plane by forcing each \( z \) coordinate to be 0? In other words, \( (x, y, z) \rightarrow (x, y, 0) \), which is called orthographic projection. If we imagine the \( xy \) plane as a virtual display of the models, then there would be several problems:

1. A jumble of objects would be superimposed, rather than hiding parts of a model that are in front of another.

Figure 3.16: Starting with any point \((x, y, z)\), a line through the origin can be formed using a parameter \( \lambda \). It is the set of all points of the form \((\lambda x, \lambda y, \lambda z)\) for any real value \( \lambda \). For example, \( \lambda = 1/2 \) corresponds to the midpoint between \((x, y, z)\) and \((0, 0, 0)\) along the line.

2. The display would extend infinitely in all directions (except \( z \)). If the display is a small rectangle in the \( xy \) plane, then the model parts that are outside of its range can be eliminated.

3. Objects that are closer should appear larger than those further away. This happens in the real world. Recall from Section 1.3 (Figure 1.19(c)) paintings that correctly handle perspective.

The first two problems are important graphics operations that are deferred until Chapter 7. The third problem is addressed next.

**Perspective projection** Instead of using orthographic projection, we define a perspective projection. For each point \((x, y, z)\), consider a line through the origin. This is the set of all points with coordinates

\[
(\lambda x, \lambda y, \lambda z),
\]

(3.38)

in which \( \lambda \) can be any real number. In other words \( \lambda \) is a parameter that reaches all points on the line that contains both \((x, y, z)\) and \((0, 0, 0)\). See Figure 3.16.

Now we can place a planar “movie screen” anywhere in the virtual world and see where all of the lines pierce it. To keep the math simple, we pick the \( z = -1 \) plane to place our virtual screen directly in front of the eye. Using the third component of (3.38), we have \( \lambda z = -1 \), implying that \( \lambda = -1/z \). Using the first two components of (3.38), the coordinates for the points on the screen are calculated as \( x' = -x/z \) and \( y' = -y/z \). Note that since \( x \) and \( y \) are scaled by the same amount \( z \) for each axis, their aspect ratio is preserved on the screen.

More generally, suppose the vertical screen is placed some location \( d \) along the \( z \) axis. In this case, we obtain more general expressions for the location of a point.
This section links all of the transformations of this chapter together while also slightly adjusting their form to match what is currently used in the VR and computer graphics industries. Some of the matrices appearing in this section may seem unnecessarily complicated. The reason is that the expressions are motivated by algorithm and hardware issues, rather than mathematical simplicity. In particular, there is a bias toward putting every transformation into a 4 by 4 homogeneous transform matrix, even in the case of perspective projection which is not even linear (recall (3.39)). In this way, an efficient matrix multiplication algorithm can be iterated over the chain of matrices to produce the result.

3.5 Chaining the Transformations

This section links all of the transformations of this chapter together while also slightly adjusting their form to match what is currently used in the VR and computer graphics industries. Some of the matrices appearing in this section may seem unnecessarily complicated. The reason is that the expressions are motivated by algorithm and hardware issues, rather than mathematical simplicity. In particular, there is a bias toward putting every transformation into a 4 by 4 homogeneous transform matrix, even in the case of perspective projection which is not even linear (recall (3.39)). In this way, an efficient matrix multiplication algorithm can be iterated over the chain of matrices to produce the result.

When $T$ is applied to a point $(x, y, z, 1)$, the location of the point on the screen is produced. Remember that these matrix multiplications are not commutative, and the operations are applied from right to left: The first matrix $T_{eb}$ is the rigid body transform (3.21) applied to points on a movable model. For each rigid object in the model, $T_{eb}$ remains the same; however, different objects will generally be placed in various positions and orientations. For example, the wheel of a virtual car will move differently than the avatar’s head. After $T_{eb}$ applied, $T_{eye}$ transforms the virtual world into the coordinate frame of the eye, according to (3.35). At a fixed instant in time, this and all remaining transformation matrices are the same for all points in the virtual world. Here we assume that the eye is positioned at the midpoint between the two virtual human eyes, leading to a cyclopean viewpoint. Later in this section, we will extend it to the case of left and right eyes so that stereo viewpoints can be constructed.

Canonical view transform The next transformation, $T_{can}$ performs the perspective projection as described in Section 3.4; however, we must explain how it is unnaturally forced into a 4 by 4 matrix. We also want to result to be in a canonical form that appears to be unitless, which is again motivated by industrial needs. Therefore, $T_{can}$ is called the canonical view transform. Figure 3.18 shows a viewing frustum, which is based on the four corners a rectangular virtual screen. At $z = n$ and $z = f$ lie a near plane and far plane, respectively. Note that $z < 0$ for these cases because the $z$ axis points in the opposite direction. The virtual screen is contained in the near plane. The perspective projection should place all of the points inside of the frustum onto a virtual screen that is centered in the near plane. This implies $d = n$ using (3.39).

We now want to try to reproduce (3.39) using a matrix. Consider the result of
applying the following matrix multiplication:
\[
\begin{bmatrix}
  n & 0 & 0 & 0 \\
  0 & n & 0 & 0 \\
  0 & 0 & n & 0 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  1 \\
\end{bmatrix}
= \begin{bmatrix}
  nx \\
  ny \\
  nz \\
  z \\
\end{bmatrix}.
\] (3.41)

In the first two coordinates, we obtain the numerator of (3.39). The nonlinear part of (3.39) is the $1/z$ factor. To handle this, the fourth coordinate is used to represent $z$, rather than 1 as in the case of $T_b$. From this point onward, the resulting 4D vector is interpreted as a 3D vector that is scaled by dividing out its fourth component. For example, $(v_1, v_2, v_3, v_4)$ is interpreted as
\[
(v_1/v_4, v_2/v_4, v_3/v_4).
\] (3.42)

Thus, the result from (3.41) is interpreted as
\[
(nz/z, ny/z, n),
\] in which the first two coordinates match (3.41) with $d = n$, and the third coordinate is the location of the virtual screen along the $z$ axis.

**Keeping track of depth for later use** The following matrix is commonly used in computer graphics, and will be used here in our chain:
\[
T_p = \begin{bmatrix}
  n & 0 & 0 & 0 \\
  0 & n & 0 & 0 \\
  0 & 0 & n + f & -fn \\
  0 & 0 & 1 & 0 \\
\end{bmatrix}.
\] (3.44)

It is identical to the matrix in (3.41) except in how it transforms the $z$ coordinate. For purposes of placing points on the virtual screen, it is unnecessary because we already know they are all placed at $z = n$. The $z$ coordinate is therefore co-opted for another purpose: Keeping track of the distance of each point from the eye so that graphics algorithms can determine which objects are in front of other objects. The matrix $T_p$ calculates the third coordinate as
\[
(n + f)z - fn.
\] (3.45)

When divided by another $z$, (3.45) does not preserve the exact distance, but the graphics methods (some of which are covered in Chapter 7) require only that the distance ordering is preserved. In other words, if point $p$ is further from the eye than point $q$, then it remains further after the transformation, even if the distances are distorted. It does, however, preserve the distance in two special cases: $z = n$ and $z = f$. This can be seen by substituting these into (3.45) and dividing by $z$.

**Additional translation and scaling** After $T_p$ is applied, the 8 corners of the frustum are transformed into the corners of a rectangular box, shown in Figure 3.19. The following performs a simple translation of the box along the $z$ axis and some rescaling so that it is centered at the origin and the coordinates of its corners are $(\pm 1, \pm 1, \pm 1)$:
\[
T_d = \begin{bmatrix}
  z/2 & 0 & 0 & -z/2 \\
  0 & t/2 & 0 & -t/2 \\
  0 & 0 & n/2 & -n/2 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}.
\] (3.46)

If the frustum is perfectly centered in the $xy$ plane, then the first two components of the last column become 0. Finally, we define the canonical view transform $T_{can}$ from (3.40) as
\[
T_{can} = T_d T_p.
\] (3.47)

**Viewport transform** The last transform to be applied in the chain (3.40) is the viewport transform $T_vp$. After $T_{can}$ has been applied, the $x$ and $y$ coordinates of each pixel range from $-1$ to 1. One last step is required to bring the projected points to the coordinates used to index pixels on a physical display. Let $m$ be the number of horizontal pixels and $n$ be the number of vertical pixels. For example, $n = 1080$ and $m = 1920$ for a 1080p display. Suppose that the display is indexed with rows running from 0 to $n$ and columns from 0 to $m$. Furthermore, $(0, 0)$ is in the lower left corner. In this case, the viewport transform is
\[
T_{vp} = \begin{bmatrix}
  \frac{m}{2} & 0 & 0 & \frac{m-1}{2} \\
  0 & \frac{n}{2} & 0 & \frac{n-1}{2} \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}.
\] (3.48)

**Left and right eyes** We now address how the transformation chain (3.40) is altered for stereoscopic viewing. Let $t$ denote the distance between the left and
right eyes. Its value in the real world varies across people, and its average is around $t = 0.064$ meters. To handle the left eye view, we need to simply shift the cyclopean (center) eye horizontally to the left. Recall from Section 3.4 that the inverse actually gets applied. The models need to be shifted to the right. Therefore, let

$$T_{\text{left}} = \begin{bmatrix} 1 & 0 & 0 & \frac{t}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

(3.49)

which corresponds to a right shift of the models, when viewed from the eye. This transform is placed after $T_{\text{eye}}$ to adjust its output. The appropriate modification to (3.40) is:

$$T = T_{\text{vp}}T_{\text{can}}T_{\text{left}}T_{\text{eye}}T_{\text{rb}}.$$  

(3.50)

By symmetry, the right eye is similarly handled by replacing $T_{\text{left}}$ in (3.50) with

$$T_{\text{right}} = \begin{bmatrix} 1 & 0 & 0 & -\frac{t}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  

(3.51)

This concludes the explanation of the entire chain of transformations to place and move models in the virtual world and then have them appear in the right place on a display. After reading Chapter 4, it will become clear that one final transformation may be needed after the entire chain has been applied. This is done to compensate for nonlinear optical distortions that occur due to wide-angle lenses in VR headsets.

**Further Reading**

References on transforming chains of bodies ([29] Chapter 3), and animating articulated structures.

The fact that mesh orientations cannot be consistently labeled for some surfaces is the basis of homology! Should include some topology references.

Euler angle references.

Need quaternion algebra references, more conversions, and derivations of all the given conversions.
Chapter 4

Light and Optics

4.1 Basic Behavior of Light

Light can be described in three ways that appear to be mutually incompatible:

1. Photons: Tiny particles of energy moving through space at high speeds (no need for quantum mechanics in this book!). This interpretation is helpful when considering the amount of light received by a sensor or receptor.

2. Waves: Ripples through space that are similar to waves propagating on the surface of water, but are 3D. The wavelength is the distance between peaks. This interpretation is helpful when considering the spectrum of colors.

3. Rays: A ray traces the motion of a single hypothetical photon. The direction is perpendicular to the wavefronts (see Figure 4.1). This interpretation is helpful when explaining lenses and defining the concept of visibility.

Fortunately, modern physics has explained how these interpretations are in fact compatible; each is useful in this book.

**Spreading waves** Figure 4.1 shows how waves would propagate from a hypothetical point light source. The density would be the same in all directions (radial symmetry), but would decrease as the light source becomes more distant. Recall that the surface area of a sphere with radius $r$ is $4\pi r^2$. Consider centering a spherical screen around the light source. The total number of photons per second hitting a screen of radius 1 should be the same as for a screen of radius 2; however, the density (photons per second per area) should decrease by a factor of 1/4 because they are distributed over 4 times the area. Thus, photon density decreases quadratically as a function of distance from a point light source.

The curvature of the wavefronts also decreases as the point light source becomes further away. If the waves were to propagate infinitely far away, then they would completely flatten as shown in Figure 4.2. This results in the important case...
4.1. BASIC BEHAVIOR OF LIGHT

If the point light source were “infinitely far” away, then parallel wavefronts would be obtained. Other names for this setting are: Collimated light, parallel rays, rays from infinity, rays to infinity, and zero vergence.

of parallel wavefronts. Without the help of lenses or mirrors, it is impossible to actually obtain this case from a tiny light source in the physical world because it cannot be so far away; however, it serves as both a useful approximation for distant light sources and as an ideal way to describe lenses mathematically. Keep in mind that at any finite distance from a point light source, the rays of light always diverge; it is impossible to make them converge without the help of lenses or mirrors.

Interactions with materials As light strikes the surface of a material, one of three behaviors might occur, as shown in Figure 4.3. In the case of transmission, the energy travels through the material and exits the other side. For a transparent material, such as glass, the transmitted light rays are slowed down and bend according to Snell’s law, which will be covered in Section 4.2. For a translucent material that is not transparent, the rays scatter into various directions before exiting. In the case of absorption, energy is absorbed by the material as the light becomes trapped. The third case is reflection, in which the light is deflected from the surface. Along a perfectly smooth or polished surface, the rays reflect in the same way: The exit angle is equal to the entry angle. See Figure 4.4. This case is called specular reflection, in contrast to diffuse reflection, in which the reflected rays scatter in arbitrary directions. Usually, all three cases of transmission, absorption, and reflection occur simultaneously. The amount of energy divided between the cases depends on many factors, such as the angle of approach, the wavelength, and differences between the materials.

A jumble of wavelengths Figure 4.1 presented an oversimplified view that will make it easy to understand idealized lenses in Section 4.2. Unfortunately, it misses many details that become important in other settings, such as understanding lens aberrations (Section 4.3) or how light interacts with materials in the physical world.
4.1. BASIC BEHAVIOR OF LIGHT

Figure 4.5: Visible light spectrum corresponds to the range of electromagnetic waves that have wavelengths between 400nm and 700nm. (Figure by David Eccles for Wikipedia.)

The remainder of this section therefore considers various realistic complications that arise.

Coherent versus jumbled light The first complication is that light sources usually do not emit coherent light, a term that means the wavefronts are perfectly aligned in time and space. A laser is an exceptional case that indeed produces coherent light. It emits parallel waves of a constant wavelength that are also synchronized in time so that their peaks align as they propagate. Common light sources, such as light bulbs and the sun, instead emit a jumble of waves that have various wavelengths and do not have their peaks aligned.

Wavelengths and colors To make sense out of the jumble of waves, we will describe how they are distributed in terms of wavelengths. Figure 4.5 shows the range of wavelengths that are visible to humans. Each wavelength corresponds to a spectral color, which is what we would perceive with a coherent light source fixed at that wavelength alone. Wavelengths between 700 and 1000nm are called infrared, which are not visible to us, but our cameras can sense them (see Section 9.2). Wavelengths between 100 and 400nm are called ultraviolet; they are not part of our visible spectrum, but some birds, insects, and fish can perceive ultraviolet wavelengths over 300nm. Thus, our notion of visible light is already tied to human perception.

Spectral power Figure 4.6 shows how the wavelengths are distributed for common light sources. An ideal light source would have all visible wavelengths represented with equal energy, leading to idealized white light. The opposite is total darkness, which is black. We usually do not allow a light source to propagate light directly onto our retinas (don’t stare at the sun!). Instead, we observe light that is reflected from objects all around us, causing us to perceive their color. Each surface has its own distribution of wavelengths that it reflects. The fraction of light energy that is reflected back depends on the wavelength, leading to the plots shown in Figure 4.7. For us to perceive an object surface as red, the red wavelengths must be included in the light source and the surface must strongly

Figure 4.6: The spectral power distribution for some common light sources. (Figure from [44]).

Figure 4.7: The spectral reflection function of some common familiar materials. (Figure from [44]).
4.2 LENSES

Lenses have been made for thousands of years, with the oldest known artifact shown in Figure 4.8(a). It was constructed before 700 BC in Assyrian Nimrud, which was coincidentally mentioned in Figure 1.15 of Chapter 1. Whether constructed from transparent materials or from polished surfaces that act as mirrors, lenses bend rays of light so that a focused image is formed. Over the centuries, their uses have given rise to several well-known devices, such as eyeglasses (Figure 4.8(b)), telescopes, magnifying glasses, binoculars, cameras, and microscopes. Optical engineering is therefore filled with design patterns that indicate how to optimize the designs of these well-designed devices. VR headsets are a newcomer among existing optical devices, leading to many new challenges that are outside of standard patterns that have existed for centuries. Thus, the lens design patterns for VR are still being written. To first step toward addressing the current challenges is to understand how simple lenses work.

Snell’s Law

Lenses work because of Snell’s Law, which expresses how much rays of light bend when entering and exiting a transparent material. Recall that the speed of light in a vacuum is a universal constant \(c\) with value approximately equal to \(3 \times 10^8\) m/s. In this case, \(s = c\) in (4.1). Light propagates roughly 0.03 percent faster in a vacuum than in air, causing the difference to be neglected in most engineering calculations. Visible light in air has a frequency range of roughly 400 to 800 terahertz, which is obtained by applying (4.1). As light propagates through denser media, such as water or lenses, \(s\) is significantly smaller; that difference is the basis of optical systems, which are covered next.

**Frequency**

Often times, it is useful to talk about frequency instead of wavelength. The frequency \(f\) is the number of times per second that wave peaks pass through a fixed location. Using both the wavelength \(\lambda\) and the speed \(s\), the frequency \(f\) is calculated as:

\[
f = \frac{s}{\lambda}. \tag{4.1}\]

The speed of light in a vacuum is a universal constant \(c\) with value approximately equal to \(3 \times 10^8\) m/s. This case, \(s = c\) in (4.1). Light propagates roughly 0.03 percent faster in a vacuum than in air, causing the difference to be neglected in most engineering calculations. Visible light in air has a frequency range of roughly 400 to 800 terahertz, which is obtained by applying (4.1). As light propagates through denser media, such as water or lenses, \(s\) is significantly smaller; that difference is the basis of optical systems, which are covered next.

\[
\sin \theta_1 = n_2 \sin \theta_2. \tag{4.3}\]

Typically, \(n_1 > n_2\) and \(\theta_1\) are given, so that (4.3) is solved for \(\theta_2\) to obtain

\[
\theta_2 = \sin^{-1} \left( \frac{n_1 \sin \theta_1}{n_2} \right). \tag{4.4}\]

If \(n_1 < n_2\), then \(\theta_2\) is closer to perpendicular than \(\theta_1\). If \(n_2 > n_1\), then \(\theta_2\) is further from perpendicular. The case of \(n_1 > n_2\) is also interesting in that light may not penetrate the surface if the incoming angle \(\theta_1\) is too large. The range of \(\sin^{-1}\) is 0 to 1, which implies that (4.4) provides a solution for \(\theta_2\) only if \((n_1/n_2) \sin \theta_1 \leq 1\).

If the condition does not hold, then the light rays always reflect from the surface. This situation occurs while under water and looking up at the surface. Rather than being able to see the world above, you might instead see a reflection, depending on the viewing angle.
**4.2. LENSES**

![Image](image.png)

Figure 4.9: Propagating wavefronts from a medium with low refractive index (such as air) to one with a higher index (such as glass). (a) The effect of slower propagation on the wavefronts is shown as they enter the lower medium. (b) This shows the resulting bending of a light ray, which is always perpendicular to the wavefronts. Snell’s Law relates the refractive indices and angles as $n_1 \sin \theta_1 = n_2 \sin \theta_2$.

**Prisms** Imagine shining a laser beam through a prism, as shown in Figure 4.10. Snell’s Law can be applied to calculate how the light ray bends after it enters and exits the prism. Note that for the upright prism, a ray pointing slightly upward becomes bent downward. Recall that a larger refractive index inside the prism would cause greater bending. By placing the prism upside down, rays pointing slightly downward are bent upward. Once the refractive index is fixed, the bending depends only on the angles at which the rays enter and exist the surface, rather than the thickness of the prism. To construct a lens, we will exploit this principle and construct a kind of curved version of Figure 4.10.

**Simple convex lens** Figure 4.11 shows a simple convex lens, which should remind you of the prisms in Figure 4.10. Instead of making a diamond shape, the lens surface is spherically curved so that incoming, parallel, horizontal rays of light converge to a point on the other side of the lens. This special place of convergence is called the focal point. Its distance from the lens center is called the focal depth or focal length.

The incoming rays in Figure 4.11 are special in two ways: 1) They are parallel, thereby corresponding to a source that is infinitely far away, and 2) they are perpendicular to the plane in which the lens is centered. If the rays are parallel but not perpendicular to the lens plane, then the focal point shifts accordingly, as shown in Figure 4.12. In this case, the focal point is not on the optical axis. There are two DOFs of incoming ray directions, leading to a focal plane that contains all of the focal points. Unfortunately, this planarity is just an approximation; Section

![Image](image.png)

Figure 4.10: The upper part shows how a simple prism bends ascending rays into descending rays, provided that the incoming ray slope is not too high. This was achieved by applying Snell’s Law at the incoming and outgoing boundaries. Placing the prism upside down causes descending rays to become ascending. Putting both of these together, we will see that a lens is like a stack of prisms that force diverging rays to converge through the power of refraction.

![Image](image.png)

Figure 4.11: A simple convex lens causes parallel rays to converge at the focal point. The dashed line is the optical axis, which is perpendicular to the lens and pokes through its center.
4.2. LENSES

Figure 4.12: If the rays are not perpendicular to the lens, then the focal point is shifted away from the optical axis.

4.3 explains what really happens. In this idealized setting, a real image is formed in the image plane, as if it were a projection screen that is showing how the world looks in front of the lens (assuming everything in the world is very far away).

If the rays are not parallel, then it may still be possible to focus them into a real image, as shown in Figure 4.13. Suppose that a lens is given that has focal length \( f \). If the light source is placed at distance \( s_1 \) from the lens, then the rays from that will be in focus if and only if the following equation is satisfied:

\[
\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{f}.
\]  

Figure 4.11 corresponds to the idealized case in which \( s_1 = \infty \), for which solving (4.5) yields \( s_2 = f \). What if the object being viewed is not completely flat and lying in a plane perpendicular to the lens? In this case, there does not exist a single plane behind the lens that would bring the entire object into focus. We must tolerate the fact that most of it will be approximately in focus. Unfortunately, this is the situation almost always encountered in the real world, including the focus provided by our own eyes (see Section 4.4).

If the light source is placed too close to the lens, then the outgoing rays might be diverging so much that the lens cannot force them to converge. If \( s_1 = f \), then the outgoing rays would be parallel (\( s_2 = \infty \)). If \( s_1 < f \), then (4.5) yields \( s_2 < 0 \).

Figure 4.13: In the real world, an object is not infinitely far away. When placed at distance \( s_1 \) from the lens, a real image forms in a focal plane at distance \( s_2 > f \) behind the lens, as calculated using (4.5).

In this case, a real image is not formed; however, something interesting happens: The phenomenon of magnification. A virtual image appears when looking into the lens, as shown in Figure 4.14. This exactly what happens when the lens is used in the View-Master and the VR headsets that were shown in Figure 2.11. The screen is placed so that it appears magnified. To the user viewing through the screen, it appears as if the screen is infinitely far away (and quite enormous!).

**Lensmaker’s equation** For a given simple lens, the focal length \( f \) can be calculated using the Lensmaker’s Equation,

\[
(n_2 - n_1) \left( \frac{1}{r_1} + \frac{1}{r_2} \right) = \frac{1}{f},
\]  

where \( r_1 \) and \( r_2 \) are the curvatures of the lens surfaces and \( n_1 \) and \( n_2 \) are the indices of refraction of the lens material and the medium on the opposite side, respectively.
which is derived from Snell’s law \[]. The parameters \( r_1 \) and \( r_2 \) represent the radius of curvature of each of the two lens surfaces (front and back). This version assumes a \textit{thin lens approximation}, which means that the lens thickness is small relative to \( r_1 \) and \( r_2 \). Also, it is typically assumed that \( n_1 = 1 \), which is approximately true for air.

**Concave lenses** For the sake of completeness, we include the case of a \textit{concave} simple lens, shown in Figure 4.15. Parallel rays are forced to diverge, rather than converge; however, a meaningful notion of negative focal length exists by tracing the diverging rays backwards through the lens. The Lensmaker’s Equation (4.6) can be slightly adapted to calculate negative \( f \) in this case \[].

**Dioptrers** For optical systems used in VR, several lenses will be combined in succession. What is the effect of the combination? A convenient method to answer this question with simple arithmetic was invented by ophthalmologists. The idea is to define a \textit{diopter}, which is \( D = 1/f \). Thus, it is the reciprocal of the focal length. If a lens focuses parallel rays at a distance of 0.2 m behind the lens, then \( D = 5 \). A larger dioptr \( D \) means greater converging power. Likewise, a concave lens yields \( D < 0 \) by using a negative lens. To combine several lenses in succession, we simply add their dipters to determine their equivalent power as a single, simple lens. Figure 4.16 shows a simple example.

### 4.3 Optical Aberrations

If lenses in the real world behaved \textit{exactly} as described in Section 4.2, then VR systems would be much simpler and more impressive than they are today. Unfortunately, numerous imperfections, called \textit{aberrations}, degrade the images formed by lenses. Because these problems are perceptible in everyday uses, such as viewing content through VR headsets or images from cameras, they are important to understand so that some compensation for them can be designed into the VR system or content.

**Chromatic aberration** Recall from Section 4.1 that light energy is usually a jumble of waves with a spectrum of wavelengths. You have probably seen that the colors of the entire visible spectrum nicely separate when white light is shined through a prism. This is a beautiful phenomenon, but for lenses it is terrible annoyance because it separates the focused image based on color. This problem is called \textit{chromatic aberration}.

The problem is that the speed of light through a medium depends on the wavelength. We therefore should write a material’s refractive index as \( n(\lambda) \) to indicate that it is a function of \( \lambda \). Figure 4.17 shows the effect on a simple convex lens. The focal depth becomes a function of wavelength. If we shine red, green, and blue lasers directly into the lens along the same ray, then each color would cross the optical axis in a different place, resulting in red, green, and blue focal points. Recall the spectral power distribution and reflection functions from Section 4.1. For common light sources and materials, the light passing through a lens results in a whole continuum of focal points. Figure 4.18 shows an image with chromatic aberration artifacts. Chromatic aberration can be reduced at greater expense by combining convex and concave lenses of different materials so that the spreading rays are partly coerced into converging \[46\].

**Spherical aberration** Figure 4.19 shows \textit{spherical aberration}, which is caused by rays further away from the lens center being refracted more than rays near the center. The result is similar to that of chromatic aberration, but this phenomenon is a \textit{monochromatic} aberration because it is independent of the light wavelength. Incoming parallel rays are focused at varying depths, rather than being concentrated at a single point. The result is some blur that cannot be compensated by moving the object, lens, or image plane. Alternatively, the image might instead focus onto a curved surface, called the \textit{Petzval surface}, rather than the image plane. This aberration arises due to the spherical shape of the lens. An \textit{aspheric lens} is more complex and has non-spherical surfaces that are designed to specifically
4.3. OPTICAL ABERRATIONS

Figure 4.18: The upper image is properly focused whereas the lower image suffers from chromatic aberration. (Figure by Stan Zurek.)

Figure 4.19: Spherical aberration causes imperfect focus because rays away from the optical axis are refracted more than those at the periphery.

Figure 4.20: Common optical distortions. (a) Original images. (b) Barrel distortion. (c) Pincushion distortion. For the upper row, the grid becomes nonlinearly distorted. For lower row illustrates how circular symmetry is nevertheless maintained.

eliminate the spherical aberration and reduce other aberrations.

**Optical distortion** Even if the image itself projects onto the image plane it might be distorted at the periphery. Assuming that the lens is radially symmetric, the distortion can be described as a stretching or compression of the image that becomes increasingly severe away from the optical axis. Figure 4.20 shows how this affects the image for two opposite cases: barrel distortion and pincushion distortion. For lenses that have a wide field-of-view, the distortion is stronger, especially in the extreme case of a fish-eyed lens. Figure 4.21 shows an image that has strong barrel distortion. Correcting this distortion is an important component of VR headsets; otherwise, the virtual world would appear to be warped.

**Astigmatism** Figure 4.22 depicts astigmatism, which is a lens aberration that occurs for incoming rays that are not perpendicular to the lens. Up until now, our lens drawings have been 2D; however, a third dimension is needed to understand this new aberration. The rays can be off-axis in one dimension, but aligned in another. By moving the image plane along the optical axis, it becomes impossible to bring the image into focus. Instead, horizontal and vertical focal depths appear,
4.3. OPTICAL ABERRATIONS

Figure 4.21: An image with barrel distortion, taken by a fish-eyed lens. (Image by Wikipedia user Ilveon.)

Figure 4.22: Astigmatism is primarily caused by incoming rays being off-axis in one plane, but close to perpendicular in another. (Figure from [50].)

Figure 4.23: Due to astigmatism, it becomes impossible to bring the image perfectly into focus. At one depth, it might be focus horizontally; while at another it is focused vertically. We are forced to chose a compromise, as shown in Figure 4.23.

Coma Finally, coma is yet another aberration. In this case, the image magnification varies dramatically as the rays are far from perpendicular to the lens. The result is a “comet” pattern in the image plane. You might have seen this while tilting a lens outside and observing bright disc patterns produced by direct sunlight. All of the aberrations of this section complicate the system or degrade the experience in a VR headset; therefore, substantial engineering effort is spent on mitigating these problems.

4.4 The Human Eye

We have covered enough concepts in this chapter to describe the basic operation of the human eye, which is clearly an important component in any VR system. Here it will be considered as part of an optical system of lenses and images. The physiological and perceptual parts of human vision are deferred until Chapter 5.

Figure 4.24 shows a cross section of the human eye facing left. Parallel light rays are shown entering from the left; compare to Figure 4.11, which showed a similar situation for an engineered convex lens. Although the eye operation is similar to the engineered setting, several important differences arise at this stage. The focal plane is replaced by a spherically curved surface called the retina. The retina contains photoreceptors that convert the light into neural pulses; this is covered in Sections 5.1 and 5.2. The interior of the eyeball is actually liquid, as opposed to air. The refractive indices of materials along the path from the outside air to the retina are shown in Figure 4.25.
4.4. THE HUMAN EYE

Figure 4.24: A simplified view of the human eye as an optical system.

Figure 4.25: A ray of light travels through five media before hitting the retina. The indices of refraction are indicated. Considering Snell's law, the greatest bending occurs due to the transition from air to the cornea. Note that once the ray enters the eye, it passes through only liquid or solid materials.

1. \( n_1 = 1.000 \) (air)
2. \( n_2 = 1.376 \) (cornea)
3. \( n_3 = 1.336 \) (aqueous fluid)
4. \( n_4 = 1.413 \) (lens)
5. \( n_5 = 1.337 \) (vitreous fluid)

The optical power of the eye The outer diameter of the eyeball is roughly 24mm, which implies that a lens of at least 40D would be required to cause convergence of parallel rays onto the retina center inside of the eye (recall dioptries from Section 4.2). There are effectively two convex lenses: The cornea and the lens. The cornea is the outermost part of the eye where the light first enters and has the greatest optical power, approximately 40D. The eye lens is less powerful and provides an additional 20D. By adding dioptries, the combined power of the cornea and lens is 60D, which means that parallel rays are focused onto the retina at a distance of roughly 17mm from the outer cornea. Figure 4.26 shows how this system acts on parallel rays for a human with normal vision. Images of far away objects are thereby focused onto the retina.

Accommodation What happens when we want to focus on a nearby object, rather than one “infinitely far” away? Without any changes to the optical system, the image would be blurry on the retina, as shown in Figure 4.27. Fortunately, and miraculously, the lens changes its diopter to accommodate the closer distance. This process is appropriately called accommodation, as is depicted in Figure 4.28. The diopter change is effected through muscles that pull on the lens to change its shape. In young children, the lens can increase its power by an additional 15 to 20D, which explains why a child might hold something right in front of your face and expect you to focus on it; they can! At 20D, this corresponds to focusing on
4.4. THE HUMAN EYE

Figure 4.28: The process of accommodation: The eye muscles pull on the lens, causing it to increase the total optical power and focus the image on the retina.

Figure 4.29: Placing a convex lens in front of the eye is another way to increase the optical power so that nearby objects can be brought into focus by the eye. This is the principle of reading glasses.

Figure 4.30: A simple VR headset by placing a screen very close to the eyes. Young adults would already be unable to bring it into focus if it were closer than 10cm. We want to bring it close so that it fills the view of the user. Therefore, the optical power is increased by using a convex lens, functioning in the same way as reading glasses. See Figure 4.30. This is also the process of magnification, from Section 4.2. The lens is placed at the distance of its focal depth. Using (4.5), this implies that $s_2 = -f$, resulting in $s_1 = \infty$. The screen appears as an enormous virtual image that is infinitely far away. Note, however, that a real image is nevertheless projected onto the retina. We do not perceive the world around us unless real images are formed on our retinas!

Vision abnormalities. The situations presented so far represent normal vision throughout a person’s lifetime. One problem could be that the optical system simply does not have enough optical power to converge parallel rays onto the retina. This condition is called hyperopia or farsightedness. Eyeglasses come to the rescue. The simple fix is to place a convex lens (positive diopter) in front of the eye. In the opposite direction, some eyes have too much optical power. This case is called myopia or nearsightedness, and a concave lens (negative diopter) is placed in front of the eye to reduce the optical power appropriately. Recall that we have two eyes, not one. This allows the possibility for each eye to have a different problem, resulting in different lens diopters per eye. Other vision problems may exist beyond optical power. The most common is astigmatism, which was covered in Section 4.3. In human eyes this is caused by the eyeball having an excessively elliptical shape, rather than being perfectly spherical. Specialized, non-simple lenses are needed to correct this condition. You might also wonder whether the aberrations from Section 4.3 occur in the human eye. They do, however, the problems, such as chromatic aberration, are corrected automatically by our brains because we have learned to interpret such flawed images our entire lives.

A simple VR headset. Now suppose we are constructing a VR headset by placing a screen very close to the eyes. Young adults already lose this ability and can accommodate up to about 10D. Thus, with normal vision they can read a book down to a distance of about 10cm (with some eye strain). Once adults reach 50 years old, little or no accommodation ability remains. This condition is called presbyopia. Figure 4.29 shows the most common treatment, which is to place reading glasses in front of the eye.

An object that is only 50cm from the cornea. Young adults already lose this ability and can accommodate up to about 10D. Thus, with normal vision they can read a book down to a distance of about 10cm (with some eye strain). Once adults reach 50 years old, little or no accommodation ability remains. This condition is called presbyopia. Figure 4.29 shows the most common treatment, which is to place reading glasses in front of the eye.
4.5. CAMERAS

Figure 4.30: In VR headsets, the lens is placed so that the screen appears to be infinitely far away.

would be required for each eye. Furthermore, if they have an astigmatism, then it cannot be corrected. Placing eyeglasses inside of the headset may be the only remaining solution, but it may be uncomfortable and will reduce the field of view.

Many details have been skipped or dramatically simplified in this section. One important detail for a VR headset is each lens should be centered perfectly in front of the cornea. If the distance between the two lenses is permanently fixed, then this is impossible to achieve for everyone who uses the headset. The interpupillary distance, or IPD, is the distance between human eye centers. The average among humans is around 64mm, but it varies greatly by race, gender, and age (in the case of children). To be able to center the lenses for everyone, the distance between lens centers should be adjustable from around 55 to 75mm. This is a common range for binoculars. Unfortunately, the situation is not even this simple because our eyes also rotate within their sockets, which changes the position and orientation of the cornea with respect to the lens. This amplifies optical aberration problems that were covered in Section 4.3. Eye movements will be covered in Section 5.3. Another important detail is the fidelity of our vision: What pixel density is needed for the screen that is placed in front of our eyes so that we do not notice the pixels? A similar question is how many dots-per-inch (DPI) are needed on a printed piece of paper so that we do not see the dots, even when viewed under a magnifying glass? We return to this question in Section 5.1.

Figure 4.31: A pinhole camera that is recommended for viewing a solar eclipse. (Figure from TimeAndDate.com.)

Figure 4.32: (a) A CMOS active-pixel image sensor. (b) A low-cost CMOS camera module (SEN-11745), ready for hobbyist projects.
4.5 Cameras

Now that we have covered the human eye, it seems natural to describe an engineered eye, otherwise known as a camera. People have built and used cameras for hundreds of years, starting with a camera obscura that allows light to pass through a pinhole and onto a surface that contains the real image. Figure 4.31 shows an example that you might have constructed to view a solar eclipse. (Recall the perspective transformation math from Section 3.4.) Eighteenth-century artists incorporated a mirror and tracing paper to un-invert the image and allow it to be perfectly copied. Across the 19th century, various chemically based technologies were developed to etch the image automatically from the photons hitting the imaging surface. Across the 20th century, film was in widespread use, until digital cameras avoided the etching process altogether by electronically capturing the image using a sensor. Two popular technologies have been a Charge-Coupled Device (CCD) array and a CMOS active-pixel image sensor, which is shown in Figure 4.32(a). Such digital technologies record the amount of light hitting each pixel location along the image, which directly produces a captured image. The costs of these devices has plummeted in recent years, allowing hobbyists to buy a camera module such as the one shown in Figure 4.32(b) for under $30 US.

**Shutters** Several practical issues arise when capturing digital images. The image is an 2D array of pixels, each of which having red (R), green (G), and blue (B) values that typically range from 0 to 255. Consider the total amount of light energy that hits the image plane. For a higher-resolution camera, there will generally be less photons per pixel because the pixels are smaller. Each sensing element (one per color per pixel) can be imagined as a bucket that collects photons, much like drops of rain. To control the amount of photons, a shutter blocks all the light, opens for a fixed interval of time, and then closes again. For a long interval (low shutter speed), more light is collected; however, the drawbacks are that moving objects in the scene will become blurry and that the sensing elements could become saturated with too much light. Photographers must strike a balance when determining the shutter speed to account for the amount of light in the scene, the sensitivity of the sensing elements, and the motion of the camera and objects in the scene.

Also relating to shutters, CMOS sensors unfortunately work by sending out the image information sequentially, line-by-line. The sensor is therefore coupled with a rolling shutter, which allows light to enter for each line, just before the information is sent. This means that the capture is not synchronized over the the entire image, which leads to odd artifacts, such as the one shown in Figure 4.33. Image processing algorithms that work with rolling shutters and motion typically transform the image to correct for this problem. CCD sensors grab and send the entire image at once, resulting in a global shutter. Unfortunately, CCDs are more expensive than CMOS sensors, which has resulted in widespread appearance of rolling shutter cameras in smartphones.

![Figure 4.33: The wings of a flying helicopter are apparently bent backwards due to the rolling shutter effect.](image)

![Figure 4.34: A spectrum of aperture settings, which control the amount of light that enters the lens. The values shown are called the focal ratio or f-stop.](image)
Aperture  The optical system also impacts the amount of light. Using a pinhole, as shown in Figure 4.31, light would fall onto the image sensor, but it would not be bright enough for most purposes (other than viewing a solar eclipse). Therefore, a convex lens is used instead so that multiple rays are converged to the same point in the image plane; recall Figure 4.11. This generates more photons per sensing element. The main drawback is that the lens sharply focuses objects at a single depth, while blurring others; recall (4.5). In the pinhole case, all depths are essentially “in focus”, but there might not be enough light. Photographers therefore want to tune the optical system to behave more like a pinhole or more like a full lens, depending on the desired outcome. This result is a controllable aperture (Figure 4.34), which appears behind the lens and sets the size of the hole through which the light rays enter. A small radius mimics a pinhole by blocking all but the center of the lens. A large radius allows light to pass through the entire lens. Our eyes control the light levels in a similar manner by contracting or dilating our pupils. Finally, note that the larger the aperture, the more that the aberrations covered in Section 4.3 affect the imaging process.

Further Reading
A classic, popular text on optical engineering: [46].
Chapter 5
The Physiology of Human Vision

Chapter Status
Taken from Virtual Reality, S. M. LaValle
This online chapter is not the final version! Check http://msl.cs.uiuc.edu/vr/ for information on the latest draft version.
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What you perceive about the world around you is “all in your head”. After reading Chapter 4, especially Section 4.4, you should understand that the light around us forms images on our retinas that capture colors, motions, and spatial relationships in the physical world. For someone with normal vision, these captured images may appear to have perfect clarity, speed, accuracy, and resolution, while being distributed over a large field of view. However, we are being fooled. We will see in this chapter that this apparent perfection of our vision is mostly an illusion because neural structures are filling in plausible details to generate a coherent picture in our heads that is consistent with our life experiences. When building VR technology that co-opts these processes, it important to understand how they work. They were designed to do more with less, and fooling these processes with VR produces many unexpected side effects because the display technology is not a perfect replica of the surrounding world.

Section 5.1 continues where Section 4.4 left off by adding some biology of the human eye to the optical system. Most of the section is on photoreceptors, which are the “input pixels” that get paired with the “output pixels” of a digital display for VR. Section 5.2 offers a taste of neuroscience by explaining what is known about the visual information that hierarchically propagates from the photoreceptors up to the visual cortex. Section 5.3 explains how our eyes move, which serves a good purpose, but incessantly interferes with the images in our retinas. Section 5.4 concludes the chapter by applying the knowledge gained about visual physiology to determine VR display requirements, such as the screen resolution.

Figure 5.1: Physiology of the human eye. This viewpoint shows how the right eye would appear if sliced horizontally (the nose would be to the left). (From Wikipedia user Rhcastilhos.)

5.1 From the Cornea to Photoreceptors

Parts of the eye Figure 5.1 shows the physiology of a human eye. The shape is approximately spherical, with a diameter of around 24mm and only slight variation among people. The cornea is a hard, transparent surface through which light enters and provides the greatest optical power (recall from Section 4.4). The rest of the outer surface of the eye is protected by a hard, white layer called the sclera. Most of the eye interior consists of vitreous humor, which is a transparent, gelatinous mass that allows light rays to penetrate with little distortion or attenuation.

As light rays cross the cornea, they pass through a small chamber containing aqueous humour, which is another transparent, gelatinous mass. After crossing this, rays enter the lens by passing through the pupil. The size of the pupil is controlled by a disc-shaped structure called the iris, which provides an aperture that regulates the amount of light that is allowed to pass. The optical power of the lens is altered by ciliary muscles. After passing through the lens, rays pass through...
5.1. FROM THE CORNEA TO PHOTORECEPTORS

The vitreous humor and strike the retina which lines more than 180° of the inner eye boundary. Since Figure 5.1 shows a 2D cross section, the retina looks more like an arc; however, keep in mind that it is a 2D surface. Imagine it as curved counterpart to a visual display. To catch the light from the output pixels, it is lined with photoreceptors, which behave like “input pixels”. The most important part of the retina is the fovea; the highest visual acuity, which is a measure of the sharpness or clarity of vision, is provided for rays that land on it. The optic disc is a small hole in the retina through which neural pulses are transmitted outside of the eye through the optic nerve. It is on the same side of the fovea as the nose.

Photoreceptors The retina contains two kinds of photoreceptors for vision: 1) rods, which are triggered by very low levels of light, and 2) cones, which require more light and are designed to distinguish between colors. See Figure 5.2. Each human retina contains about 120 million rods and 6 million cones. Figure 5.3 shows the detection capabilities of each photoreceptor type. Rod sensitivity peaks at 498nm, between blue and green in the spectrum. There are three categories of cones, based on whether they are designed to sense blue, green, or red light.

Photoreceptors respond to light levels over a large dynamic range. Figure 5.4 shows several familiar examples. The luminance is measured in SI units of candelas per square meter, which corresponds directly to the amount of light power per area.

![Figure 5.2:](image)

![Figure 5.3:](image)

![Figure 5.4:](image)
5.1 FROM THE CORNEA TO PHOTORECEPTORS

Figure 5.5: Photoreceptor density as a function of angle. The right of the plot is the nasal side (which corresponds to rays entering from the opposite, temporal side). (Figure based on [39])

The range spans seven orders of magnitude, from 1 photon hitting a photoreceptor every 100 seconds up to 100,000 photons per receptor per second. At low light levels, only rods are triggered. Our inability to distinguish colors at night is caused by the inability of rods to distinguish colors. Our eyes may take up to 35 minutes to fully adapt to low light, resulting in a monochromatic mode called scotopic vision. By contrast, our cones become active in brighter light. Adaptation to this trichromatic mode, called photopic vision, may take up to ten minutes (you have undoubtedly noticed the adjustment period when someone unexpectedly turns on lights while you are lying in bed at night).

Photoreceptor density The density of photoreceptors across the retina varies greatly, as plotted in Figure 5.5. The most interesting region is the fovea, which has the greatest concentration of photoreceptors. The innermost part of the fovea has a diameter of only 0.5mm or an angular range of ±0.85 degrees, and contains almost entirely cones. This implies that the eye must be pointed straight at a target to perceive a sharp, colored image. The entire fovea has diameter 1.5mm (±2.6 degrees angular range), with the outer ring having a dominant concentration of rods. Rays that enter the cornea from the sides land on parts of the retina with lower rod density and very low cone density. This corresponds to the case of peripheral vision. We are much better at detecting movement in our periphery, but cannot distinguish colors effectively. Peripheral movement detection may have helped our ancestors from being eaten by predators. Finally, the most intriguing part of the plot is the blind spot, where there are no photoreceptors. This is due to our retinas being inside-out and having no other way to route the neural signals to the brain; see Section 5.2.

The photoreceptor densities shown in Figure 5.5 leave us with a conundrum. With 20/20 vision, we perceive the world as if our eyes are capturing a sharp, colorful image over a huge angular range. This seems impossible, however, because we can only sense sharp, colored images in a narrow range. Furthermore, the blind spot should place a black hole in our image. Surprisingly, our perceptual processes produce an illusion that a complete image is being captured. This is accomplished by filling in the missing details using contextual information, which is described in Section 5.2, and by frequent eye movements, the subject of Section 5.3. If you are still not convinced that your brain is fooling you into seeing a complete image, try the blind spot experiment shown in Figure 5.6.

5.2 From Photoreceptors to the Visual Cortex

Photoreceptors are transducers that convert the light-energy stimulus into an electrical signal called a neural impulse, thereby inserting information about the outside world into our neural structures. Recall from Section 2.3 that signals are propagated upward in a hierarchical manner, from photoreceptors to the visual cortex (Figure 2.19). Think about the influence that each photoreceptor has on the network of neurons. Figure 5.7 shows a simplified model. As the levels increase, the number of influenced neurons grows rapidly. Figure 5.8 shows the same diagram, but highlighted in a different way by showing how the number of photoreceptors that influence a single neuron increases with level. Neurons at the lowest levels are able to make simple comparisons of signals from neighboring photoreceptors. As the levels increase, the neurons may respond to a larger patch of the retinal image. This principle will become clear when seeing more neural structures in this section. Eventually, when signals reach the highest levels (beyond these figures), information from the memory of a lifetime of experiences is fused with the information that propagated up from photoreceptors. As the brain performs significant perceptual processing, a perceptual phenomenon results, such as recognizing a face or judging the size of a tree. It takes the brain over 100ms to produce a result that enters our consciousness.
5.2. FROM PHOTORECEPTORS TO THE VISUAL CORTEX

Figure 5.7: Four levels in a simple hierarchy are shown. Each disk corresponds to a neural cell or photoreceptor, and the arrows indicate the flow of information. Photoreceptors generate information at Level 0. In this extremely simplified and idealized view, each photoreceptor and neuron connects to exactly three others at the next level. The red and gold part highlights the growing zone of influence that a single photoreceptor can have as the levels increase.

Figure 5.8: This diagram is the same as Figure 5.7 except that the information feeding into a single neuron is highlighted. Consider the set of photoreceptors involved in the reaction of a single neural cell. This is called the receptive field. As the level increases, the receptive field size grows dramatically. Due to the spatial arrangement of the photoreceptors, this will imply that each neuron responds to a growing patch in the image on the retina. The patch increases in size at higher levels.

Figure 5.9: Light passes through a few neural layers before hitting the rods and cones. (Figure by the Institute for Dynamic Educational Advancement.)
5.2. FROM PHOTORECEPTORS TO THE VISUAL CORTEX

Figure 5.10: Vertebrates (including humans) have inside-out retinas, which lead to a blind spot and photoreceptors aimed away from the incoming light. The left shows a vertebrate eye, and the right shows a cephalopod eye, for which nature got it right: The photoreceptors face the light and there is no blind spot. (Figure by Jerry Crimson Mann.)

Now consider the first layers of neurons in more detail, as shown in Figure 5.9. The information is sent from right to left, passing from the rods and cones to the bipolar, amacrine, and horizontal cells. These three types of cells are in the inner nuclear layer. From there, the signals reach the ganglion cells, which form the ganglion cell layer. Note that the light appears to be entering from the wrong direction: It passes over these neural cells before reaching the photoreceptors. This is due to the fact that the human retina is inside-out, as shown in Figure 5.10. Evolution got it right with octopuses and other cephalopods, for which the light directly reaches the photoreceptors. One consequence of an inside-out retina is that the axons of the ganglion cells cannot be directly connected to the optic nerve (item 3 in Figure 5.10), which sends the signals outside of the eye. Therefore, a hole has been punctured in our retinas so that the “cables” from the ganglion cells can be routed outside of the eye (item 4 in Figure 5.10). This causes the blind spot that was illustrated in Figure 5.6.

Upon studying Figure 5.9 closely, it becomes clear that the neural cells are not arranged in the ideal way of Figure 5.8. The bipolar cells transmit signals from the photoreceptors to the ganglion cells. Some bipolars connect only to cones, with the number being between 1 and 10 per bipolar. Others connect only to rods, with about 30 to 50 rods per bipolar. There are two types of bipolar cells based on their function. An ON bipolar activates when the rate of photon absorption in its connected photoreceptors increases. An OFF bipolar activates for decreasing photon absorption. The bipolars connected to cones have both kinds; however, the bipolars for rods have only ON bipolars. The bipolar connections are considered to be vertical because the connect directly from photoreceptors to the ganglion cells.

This is in contrast to the remaining two cell types in the inner nuclear layer. The horizontal cells are connected by inputs (dendrites) to photoreceptors and bipolar cells within a radius of up to 1mm. Their output (axon) is fed into photoreceptors, causing lateral inhibition, which means that the activation of one photoreceptor tends to decrease the activation of its neighbors. Finally, amacrine cells connect horizontally between bipolar cells, other amacrine cells, and vertically to ganglion cells. There are dozens of types, and their function is not well understood. Thus, scientists do not have a complete understanding of human vision, even at the lowest layers. Nevertheless, the well understood parts contribute greatly to our ability to design effective VR systems and predict other human responses to visual stimuli.

At the ganglion cell layer, several kinds of cells process portions of the retinal image. Each ganglion cell has a large receptive field, which corresponds to the photoreceptors that contribute to its activation as shown in Figure 5.8. The three most common and well understood types of ganglion cells are called midget, parasol, and bistratified. They perform simple filtering operations over their receptive fields based on spatial, temporal, and spectral (color) variations in the stimulus across the photoreceptors. Figure 5.11 shows one example. In this case, a ganglion cell is triggered when red is detected in the center but not green in the surrounding area. This condition is an example of spatial opponency, for which neural struc-
5.3. EYE MOVEMENTS

Eye rotations are a complicated and integral part of human vision. They occur both voluntarily and involuntarily, and allow humans to fixate on features in the world, even as the head or target features are moving. One of the main reasons for movement is to position the feature of interest on the fovea. Recall from Section 5.2 that only the fovea can sense dense, color images, and it unfortunately spans a very narrow field of view. To gain a coherent, detailed view of a large object,

**Figure 5.11:** The receptive field of an ON-center ganglion cell. (Figure by the Institute for Dynamic Educational Advancement.)

tures are designed to detect local image variations. Thus, consider ganglion cells as tiny image processing units that can pick out local changes in time, space, and/or color. They can detect and emphasize simple image features such as edges. Once the ganglion axons leave the eye through the optic nerve, a significant amount of image processing has already been performed to aid in visual perception. The raw image based purely on photons hitting the photoreceptor never leaves the eye.

The optic nerve connects to a part of the thalamus called the lateral geniculate nucleus (LGN); see Figure 5.12. The LGN mainly serves as a router that sends signals from the senses to the brain, but also performs some processing. The LGN sends image information to the primary visual cortex (V1), which is located at the back of the brain. The visual cortex, highlighted in Figure 5.13, contains interconnected areas labeled from V1 to V6, each performing specialized functions. Figure 5.14 shows one well-studied operation performed by the visual cortex. Chapter 6 will describe visual perception, which is the conscious result of processing in the visual cortex, based on neural circuitry, stimulation of the retinas, information from other senses, and expectations based on prior experiences. Characterizing how all of these processes function and integrate together remains an active field of research.

**Figure 5.12:** The visual pathway from the eyes to the LGN to the visual cortex. Note that information from the right and left sides of the visual field becomes swapped in the cortex. (Figure from Nature Reviews: Neuroscience)
5.3. EYE MOVEMENTS

the eyes rapidly scan over it while fixating on points of interest. Figure 5.15 shows an example. Another reason for eye movement is that our photoreceptors are slow to respond to stimuli due to their chemical nature. They take up to 10ms to fully respond to stimuli and produce a response for up to 100ms. Eye movements help keep the image fixed on the same set of photoreceptors so that they can fully charge. This is similar to the image blurring problem that occurs in cameras at low light levels and slow shutter speeds. Additional reasons for eye movement are to maintain a stereoscopic view and to prevent adaptation to a constant stimulation. To support the last claim, it has been shown experimentally that when eye motions are completely suppressed, visual perception disappears completely [20]. As movements combine to build a coherent view, it is difficult for scientists to predict and explain how we will interpret some stimuli. For example, the optical illusion in Figure 5.16 appears to be moving when our eyes scan over it.

Eye muscles The rotation of each eye is controlled by six muscles that are each attached to the sclera (outer eyeball surface) by a tendon. Figures 5.17 and 5.18 show their names and arrangement. The tendons pull on the eye in opposite pairs. For example, to perform a yaw (side-to-side) rotation, the tensions on the medial rectus and lateral rectus are varied while the other muscles are largely unaffected. To cause a pitch motion, four muscles per eye become involved. All six are involved to perform both a pitch and yaw, for example, looking upward and to the right. A small amount of roll can be generated; however, our eyes are generally not designed for much roll motion. Imagine if you could turn your eyeballs upside-down inside of their sockets! Thus, it is reasonable in most cases to approximate eye rotations as a 2D set that includes only yaw and pitch, rather than the full 3 DOFs obtained.
5.3. EYE MOVEMENTS

for rigid body rotations in Section 3.2.

Types of movements We now consider movements based on their purpose, resulting in six categories: 1) saccades, 2) smooth pursuit, 3) vestibulo-ocular reflex, 4) optokinetic reflex, 5) vergence, and 6) microsaccades. All of motions cause both eyes to rotate approximately the same way, except for vergence, which causes the eyes to rotate in opposite directions. We will skip a seventh category of motion, called rapid eye movements (REMs), because they only occur while we are sleeping and therefore do not contribute to a VR experience. The remaining six categories will now be discussed in detail.

Saccades The eye can move in a rapid motion called a saccade, which lasts less than 45ms and rotates at a rate of about 900° per second. The purpose is to quickly relocate the fovea so that important features in a scene are sensed with highest visual acuity. Figure 5.15 showed an example in which a face is scanned by fixing on various features in rapid succession. Each transition between features is accomplished by a saccade. Interestingly, our brains use saccadic masking to hide the intervals of time over which saccades occur from our memory. This results in distorted time perception, as in the case when second hands click into position on an analog clock. The result of saccades is that we obtain the illusion of high acuity over a large angular range. Although saccades frequently occur while we have little or no awareness of them, we have the ability to consciously control them as we choose features for fixation.

Smooth pursuit In the case of smooth pursuit, the eye slowly rotates to track a moving target feature. Examples are a car, a tennis ball, or a person walking by. The rate of rotation usually less than 30° per second, which is much slower than for saccades. The main function of smooth pursuit is to reduce motion blur on the retina: this is also known as image stabilization. The blur is due to the slow response time of photoreceptors, as discussed in Section 5.1. If the target is moving too fast, then saccades may be intermittently inserted into the pursuit motions to catch up to it.

Vestibulo-ocular reflex One of the most important motions to understand for VR is the vestibulo-ocular reflex or VOR. Hold your finger at a comfortable distance in front of your face. Next, yaw your head back and forth (like you are nodding “no”), turning about 20 or 30 degrees to the left and right sides each time. You may notice that your eyes are effortlessly rotating to counteract the rotation of your head so that your finger remains in view. The eye motion is involuntary. If you do not believe it, then try to avoid rotating your eyes while paying attention to your finger and rotating your head. It is called a reflex because the motion control bypasses higher brain functions. Figure 5.19 shows how this circuitry works. Based on angular accelerations sensed by our vestibular organs, signals are sent to the eye muscles to provide the appropriate counter motion.
5.3. EYE MOVEMENTS

Figure 5.19: The vestibulo-ocular reflex (VOR). The eye muscles are wired to angular accelerometers in the vestibular organ to counter head movement with the opposite eye movement with less than 10ms of latency. The connection between the eyes and vestibular organ is provided by specialized vestibular and extraocular motor nuclei, thereby bypassing higher brain functions.

Figure 5.20: In the process of stereopsis, both eyes are fixated on the same feature in the world. To transition from a close to far feature, a divergence motion occurs. A convergence motion happens for the opposite transition.

The main purpose of the VOR is to provide image stabilization, as in the case of smooth pursuit. For more details about the vestibular organ, see Section 8.4.

Optokinetic reflex The next category is called the optokinetic reflex, which occurs when a fast object speeds along. This occurs when watching a fast-moving train while standing nearby on fixed ground. The eyes rapidly and involuntarily choose features for tracking on the object, while alternating between smooth pursuit and saccade motions.

Vergence Stereopsis refers to the case in which both eyes are fixated on the same object, resulting in a single perceived image. Two kinds of vergence motions occur to align the eyes with an object. See Figure 5.20. If the object is closer than a previous fixation, then a convergence motion occurs. This means that the eyes are rotating so that the pupils are becoming closer. If the object is further, then divergence motion occurs, which causes the pupils to move further apart. The eye orientations resulting from vergence motions provide important information about the distance of objects.

Microsaccades The sixth category of movements is called microsaccades, which are small, involuntary jerks of less than one degree that trace out an erratic path. They are believed to augment many other processes, including control of fixations, reduction of perceptual fading due to adaptation, improvement of visual acuity, and resolving perceptual ambiguities [40]. Although these motions have been known since the 18th century [11], their behavior is extremely complex and not fully understood. Microsaccades are an active topic of research in perceptual psychology, biology, and neuroscience.

Eye and head movements together Although this section has focused on eye movement, it is important to understand that most of the time the eyes and head are moving together. Figure 5.21 shows the angular range for yaw rotations of the head and eyes. Although eye yaw is symmetric by allowing 35° to the left or right,
5.4 IMPLICATIONS FOR VR

Figure 5.21: The head and eyes rotate together to fixate on moving or new targets.

pitching of the eyes is not. Human eyes can pitch $20^\circ$ upward and $25^\circ$ downward, which suggests that it might be optimal to center a VR display slightly below the pupils when the eyes are looking directly forward. In the case of VOR, eye rotation is controlled to counteract head motion. In the case of smooth pursuit, the head and eyes may move together to keep a moving target in the preferred viewing area.

5.4 Implications for VR

This chapter has so far covered the human hardware for vision. Basic physiological properties, such as photoreceptor density or VOR circuitry directly impact the engineering requirements for visual display hardware. The engineered systems must be good enough to adequately fool our senses, but they need not have levels of quality that are well beyond the limits of our receptors. Thus, the VR display should ideally be designed to perfectly match the performance of the sense it is trying to fool.

How good does the VR visual display need to be? Three crucial factors for the display are:

1. Spatial resolution: How many pixels per square area are needed?
2. Intensity resolution and range: How many intensity values can be produced, and what the minimum and maximum intensity values?
3. Temporal resolution: How fast do displays need to change their pixels?

The spatial resolution factor will be addressed in the next paragraph. The second factor could also be called color resolution and range because the intensity values of each red, green, or blue subpixel produce points in the space of colors; see Section 6.3. Recall the range of intensities from Figure 5.4 that trigger photoreceptors. Photoreceptors can span seven orders of magnitude of light intensity. However, displays have only 256 intensity levels per color to cover this range. Entering scotopic vision mode does not even seem possible using current display technology because of the high intensity resolution needed at extremely low light levels. Temporal resolution is extremely important, but is deferred until Section 6.2, in the context of motion perception.
5.4. IMPLICATIONS FOR VR

Figure 5.24: Red, green, and blue cone photoreceptors are distributed in a complicated mosaic in the center of the fovea. (Figure by Mark Fairchild.)

We now address the spatial resolution. Insights into needed spatial resolution are obtained from the photoreceptor densities. As shown in Figure 5.22, we see individual lights when a display is highly magnified. As it is zoomed out, we may still perceive sharp diagonal lines as being jagged, as shown in Figure 5.23(a); this phenomenon is known as aliasing. Another artifact is the screen-door effect, shown in Figure 5.23(b); this is commonly noticed in an image produced by a digital LCD projector. What does the display pixel density need to be so that we do not perceive individual pixels? In 2010, Steve Jobs of Apple Inc. claimed that 2,326 pixels per inch (165 pixels per mm) is enough, achieving what they called a retina display.

Assume that the fovea is pointed directly at the display to provide the best sensing possible. The first issue is that red, green, and blue cones are arranged in a mosaic, as shown in Figure 5.24. The patterns are more erratic than the engineered versions in Figure 5.22. Vision scientists and neurobiologists have studied the effective input resolution through measures of visual acuity. One measure is cycles per degree, which roughly corresponds to the number of stripes that can be seen as separate along a viewing arc; see Figure 5.25. This has been estimated to be around 77 per degree, based on photoreceptor density and neural processes [10].

Supposing that one cycle is equivalent to the distance between pixel centers, this implies that $77 \times 77 = 5929$ pixels should exist over a square area of one degree by one degree.

Using simple trigonometry,

$$s = d \tan \theta,$$

we can determine what the span $s$ should be for a viewing angle $\theta$ at a distance $d$ from the eye. For very small $\theta$, $\tan \theta \approx \theta$ (in radians). If a smartphone screen is placed 30cm in front of an eye, then $s = 5.2366\text{mm}$ per degree. This means that 5929 pixels should be distributed over $5.246 \times 5.236 = 27.415\text{mm}$, which implies that the display should have at least 216.4 pixels per mm$^2$ to be a retina display.

The density can be significantly lower and nevertheless be a retina display due to optical aberrations in the human eye [23], thereby making Apple’s claim of 165 reasonable. Instead of 77 cycles per degree, 60 cycles per degree is commonly used to account for thus, which would require only 131.3 pixels per mm$^2$.

In the case of VR, we are not looking directly at a screen as in the case of smartphones. If we bring the screen up to 10cm from the eye, then $s = 1.745\text{mm}$ per degree and the required density increases (using 77 cycles per degree) to 1947 pixels per mm$^2$; however, it becomes uncomfortable or impossible for people to focus on it at such close range. By inserting a lens for magnification, the display can be brought even closer to the eye. This is commonly done for VR headsets, as was shown in Figure 4.30. Suppose the lens is positioned at its focal distance away from the screen; a typical number is 4cm. In this case, $s = 0.608\text{mm}$ and 12165 pixels per mm$^2$ would be needed. For comparison, the most dense smartphone screen available today is the Super AMOLED 1440x2560 5.1 inch screen on the Samsung S6. It has about 516 pixels per mm$^2$; using units that are common in industry, the Samsung S6 screen has 577 pixels per inch (along each axis), but the requirement for an effective retina display would be 2801 pixels per inch. Using 60 cycles per degree, it would be only 7386 pixels per mm$^2$ or 2182 pixels per inch. Thus, to approach the threshold for a VR retina display using a screen and simple lenses, the pixel density needs to be about four times higher in each dimension.

How much FOV is enough? What if the screen is brought even closer to the eye to fill more of the field of view? Based on the photoreceptor density plot in Figure 5.5 and the limits of eye rotations shown in Figure 5.21, the maximum FOV seems to be around $270^\circ$, which is larger than what could be provided by a flat screen (less than $180^\circ$). Increasing the FOV by bringing the screen closer...
would require even higher pixel density, but lens aberrations (Section 4.3) at the periphery may limit the effective field of view. Furthermore, if the lens is too thick and too close to the eye, then the eyelashes may scrape it; Fresnel lenses may provide a thin alternative, but present additional artifacts. Thus, the quest for a VR retina display may end with a balance between optical system quality and limitations of the human eye. Curved screens may help alleviate some of the problems.

**Foveated rendering: An exotic solution** One of the frustrations with this analysis is that we have not been able to exploit that fact that photoreceptor density away from the fovea is lower. We had to be conservative because we have no control over which part of the display the user will be looking at. If we could track where the eye is looking and have a tiny, movable display that is always positioned in front of the pupil, with zero delay, then much fewer pixels would be needed. This would greatly decrease computational burdens on graphical rendering systems (covered in Chapter 7). Instead of moving a tiny screen, the process can be simulated by keeping the fixed display but focusing the graphical rendering only in the spot where the eye is looking. This is called *foveated rendering*, which has been shown to work [18], but is currently too costly and there is too much delay and other discrepancies between the eye movements and the display updates. In light of microsaccades, it seems particularly daunting to match the output pixels to the photoreceptors at the highest levels of resolution and in a way the naturally mimics the real world so that the eyes benefit from the microsaccade motions.

**VOR gain adaptation** The VOR gain is a ratio that compares the eye rotation rate (numerator) to counter the rotation and translation rate of the head (denominator). Because head motion has six DOFs, it is appropriate to break the gain into six components. In the case of head pitch and yaw, the VOR gain is close to 1.0. For example, if you yaw your head to the left at 10° per second, then the eye yaws at 10° per second in the opposite direction. The VOR roll gain is very small because the eyes have a tiny roll range. The VOR translational gain depends on the distance to the features.

Recall from Section 2.3 that adaptation is a universal feature of our sensory systems. VOR gain is no exception. For those who wear eyeglasses, the VOR gain must adapt due to the optical transformations described in Section 4.2. Lenses affect the field of view and perceived size and distance of objects. The VOR comfortably adapts to this problem by changing the gain. Now suppose that you are wearing a VR headset that may suffer from flaws such as an imperfect optical system, tracking latency, and incorrectly rendered objects on the screen. In this case, adaptation may occur as the brain attempts to adapt its perception of stationarity to compensate for the flaws. In this case, your visual system could convince your brain that the headset is functioning correctly, and then your perception of stationarity in the real world would become distorted until you readapt. For example, after a flawed VR experience, you might yaw your head in the real world and have the sensation that truly stationary objects are sliding back and forth!

**Display scanout** Recall from Section 4.5 that cameras have either a rolling or global shutter based on whether the sensing elements are scanned line-by-line or in parallel. Displays work the same way, but whereas cameras are an *input* device, displays are the *output* analog. The vast majority of displays today have a rolling scanout (called *raster scan*), rather than global scanout. This implies that the pixels are updated line by line, as shown in Figure 5.26. This procedure is an artifact of old TV sets and monitors, which each had a cathode ray tube (CRT) with phosphor elements on the screen. An electron beam was bent by electromagnets so that it would repeatedly strike and refresh the glowing phosphors.

Due to the slow charge and response time of photoreceptors, we do not perceive the scanout pattern during normal use. However, when our eyes, features in the scene, or both are moving, then side effects of the rolling scanout may become perceptible. Think about the operation of a line-by-line printer, as in the case of a receipt printer on a cash register. If we pull on the tape while it is printing, then the lines would become stretched apart. If it is unable to print a single line at once, then the lines themselves would become slanted. If we could pull the tape to the side while it is printing, then the entire page would become slanted. You can also achieve this effect by repeatedly drawing a horizontal line with a pencil while using the other hand to gently pull the paper in a particular direction. The paper in this analogy is the retina and the pencil corresponds to light rays attempting to charge photoreceptors. Figure 5.27 shows how a rectangle would distort under cases of smooth pursuit and VOR. Current displays have an option called *esync*, which synchronizes output to the display so that the displayed image corresponds to a single, undistorted frame. By turning off this option, more frames per second could be rendered to the display then it can handle. The display buffer simply gets updated in the middle of the scan. This reduces the side effects due to scanout, but introduces another artifact called *tearing* (as in tearing a sheet of paper). Further

---

1This frequently happened to the author while developing and testing the Oculus Rift.
5.4. IMPLICATIONS FOR VR

improvements could be made by *racing the beam*, which means rendering each line by taking into account the precise time at which it will be drawn [4]. Yet another problem with displays is that the pixels could take so long to switch (up to 20ms) that sharp edges appear to be blurred. We will continue discussing these problems in Section 6.2 in the context of motion perception.

Retinal image slip Recall that eye movements contribute both to maintaining a target in a fixed location on the retina (smooth pursuit, VOR) and also changing its location slightly to reduce perceptual fading (microsaccades). During ordinary activities (not VR), the eyes move and the image of a feature may move slightly on the retina due to motions and optical distortions. This is called *retinal image slip*. Once a VR headset is used, the motions of image features on the retina might not match what would happen in the real world. This is due to many factors already mentioned, such as optical distortions, tracking latency, and display scanout. Thus, the retinal image slip due to VR artifacts does not match the retinal image slip encountered in the real world? The consequences of this barely been identified, much less characterized scientifically. They are likely to contribute to fatigue, and possibly VR sickness. As an example of the problem, there is evidence that microsaccades are triggered by the lack of retinal image slip [13]. This implies that differences in retinal image slip due to VR usage could interfere with microsaccade motions, which are already not fully understood.

Vergence-accommodation mismatch Recall from Section 4.4 that accommodation is the process of changing the eye lens’ optical power so that close objects can be brought into focus. This normally occurs with both eyes fixated on the same object, resulting in a stereoscopic view that is brought into focus. In the real world, the vergence motion of the eyes and the accommodation of the lens are tightly coupled. For example, if you place your finger 10cm in front of your face, then your eyes will try to increase the lens power while the eyes are strongly converging. If a lens is placed at a distance of its focal length from a screen, then with normal eyes it will always be in focus while the eye is relaxed (recall Figure 4.30). What if an object is rendered to the screen so that it appears to be only 10cm away? In this case, the eyes strongly converge, but they do not need to change the optical power of the eye lens. The eyes may nevertheless try to accommodate, which would have the effect of blurring the perceived image. The result is called *vergence-accommodation mismatch* because the stimulus provided by VR is inconsistent with the real world. Even if the eyes become accustomed to the mismatch, the user may feel extra strain or fatigue after prolonged use. The eyes are essentially being trained to allow a new degree of freedom: Separating vergence from accommodation, rather than coupling them. Engineering solutions may provide some relief from this problem, but they are currently too costly and imprecise. For example, the mismatch can be greatly reduced by employing eye tracking to estimate the amount of vergence and then altering the power of the optical system [1, 2].

Further Reading

For further reading on the photoreceptor mosaic, see Chapter 3 of [48].
Photoreceptor density variation over humans: [10].
Retina display analysis:
http://prometheus.med.utah.edu/~bwjones/2010/06/apple-retina-display/
More about eyes and lenses together: [45].
All about eye movement from a neuroscience perspective: [30].
VOR gain adaptation: [16, 42]
http://europepmc.org/abstract/med/3675487
Survey of microsaccades: [40]
Smooth pursuit and saccade coordination: [14]
Monkey paper for head/eye coordination: [28]
See Oxford Handbook of Eye Movements: [31]
For more neuroscience, see Chapter 7 of [33].
Nice vision summary:
You can also figure out whether it is worthwhile to upgrade your TV by using the retina display analysis:
http://www.rtings.com/tv/learn/size-to-distance-relationship
Abrash blog post about scanout:
http://blogs.valvesoftware.com/abrash/raster-scan-displays-more-than-meets-the-eye/
Comfort of vergence-accommodation mismatch: [43]
Chapter 6

Visual Perception

This chapter continues where Chapter 5 left off by transition from the physiology of human vision to perception. If we were computers, then this transition might seem like going from low-level hardware to higher-level software and algorithms. How do our brains interpret the world around us so effectively in spite of our limited biological hardware?

Section 6.1 covers perception of the distance of objects from our eyes, which is also related to the perception of object scale. Section 6.2 explains how we perceive motion. An important part of this is the illusion of motion that we perceive from videos, which are merely a sequence of pictures. Section 6.3 covers the perception of color, which may help explain why displays use only three colors (red, green, and blue) to simulate the entire spectral power distribution of light (recall from Section 4.1). Finally, Section 6.4 presents a statistically based model of how information, called cues, are combined from multiple sources to produce a perceptual experience.

VR itself is an elaborate optical illusion.

6.1 Perception of Depth

Important! We get MANY more cues than stereo. In fact, stereo may be more trouble than it is worth.

Depth cues may be metric or ordinal

Monocular cues:
  retinal image size

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height in visual field
texture gradients
image blur
atmospheric perspective
accommodation
motion parallax: look out the side window of a car or train.
shadows/shading
interposition
interposition
Stereo cues:
vergence angle
binocular disparity
diplopia
show horopter. compare to other animals (fish)
Combining cues: Bayesian is not a bad model
scale or size perception is related.

6.2 Perception of Motion

Neural circuitry for motion detection.

Optical flow. Vection (at least mention)?
Wagon wheel
Phi phenomenon
Framerate history
Stroboscopic apparent motion
6.2. PERCEPTION OF MOTION

Figure 6.2: Floating balls.

Figure 6.3: Interposition of objects.

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<td>2</td>
<td>Motion perception starts</td>
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<tr>
<td>10</td>
<td>Lose ability to distinguish individual frames</td>
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<tr>
<td>16</td>
<td>Old home movies</td>
</tr>
<tr>
<td>24</td>
<td>Hollywood classic standard</td>
</tr>
<tr>
<td>60</td>
<td>Standard for games; uncomfortable flicker for low persistence</td>
</tr>
<tr>
<td>72</td>
<td>Minimum CRT refresh rate for comfort</td>
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<tr>
<td>100</td>
<td>No more discomfort from flicker</td>
</tr>
<tr>
<td>1000</td>
<td>Ability to see zipper effect for fast, blinking LED</td>
</tr>
<tr>
<td>5000</td>
<td>Cannot perceive zippers</td>
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6.3 Perception of Color

Color spaces.

- Why does a display need only have R, G, B?
- Problem: Limited dynamic range of displays.

6.4 Combining Sources of Information

A Bayesian view, or statistical decision theory...

- Could be multiple cues per sense
- Could be multiple senses, as in vection
- Helps explain adaptation as more data weighs in

Further Reading
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Visual Rendering

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7.2 Shading Models

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8.2 Collision Detection
8.3 Velocity and Acceleration
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9.2 Estimating Position and Orientation

9.3 Tracking Kinematic Structures

9.4 SLAM

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Fusion of IMU and Vision for Absolute SCale Estimation, Nutzi, Scaramuzza, Weiss, Siegwart.
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10.3 System Control

10.4 Social Interaction

10.5 Specialized Interfaces
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Evaluating VR Systems

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Examples: Blind people can hear better, and deaf people can see better. Doctors and medical technicians train to read nonintuitive medical scans. Detectives train to look for clues at a crime scene. Question: What should VR engineers and developers train for?

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