# CS 475:Formal Models of Computation
# Homework 5

1. **[10 points]** Let $(\mathbb{N}, <)$ be the model with universe $\mathbb{N}$ (natural numbers) and the ”less than” relation. Prove that Th$(\mathbb{N}, <)$ is decidable.

   **Answer**. Proof is practically the same as in Sipser for Th$(\mathbb{N}, +)$, pages 227–229. The only change is to construct a DFA that verifies the $<$ relation, rather than $+$. Alternatively, use the reduction on Sipser, page 244.

2. **[10 points]** Prove that the following languages are undecidable (hint: Use Rice's theorem).

   - EMPTY$_{TM} = \{\langle M \rangle \mid M$ is a TM and $L(M) = \emptyset\}$.
     **Answer**. To use Rice's theorem we need to verify two conditions. For the first one we have that if $L(M_1) = L(M_2)$, then $M_1 \in$ EMPTY$_{TM}$ iff $M_2 \in$ EMPTY$_{TM}$. This condition is true by the construction of EMPTY$_{TM}$. For the second condition, we need to verify that there is a T.M. not in EMPTY$_{TM}$. This is trivially true, since there are machines with non-empty languages. Therefore, by Rice's theorem, EMPTY$_{TM}$ is undecidable.
   - ALL$_{TM} = \{\langle M \rangle \mid M$ is a TM and $L(M) = \Sigma^*\}$.
     **Answer**. [From the previous point, substitute EMPTY$_{TM}$ with ALL$_{TM}$, and “with non-empty languages” with “that reject some string”].

3. **[10 points]** Describe two different Turing Machines, $M$, and $N$, that, when started on any input, $M$ outputs $\langle N \rangle$, and $N$ outputs $\langle M \rangle$.

   **Answer.** To solve this problem, we have to be careful about circular definitions, that is, defining $M$ in terms of $N$, and $N$ in terms of $M$. Remember that $q(w)$ is a T.M. that prints $w$ on its tape, and halts.

   M = ” On input $w$:

   (a) Ignore input.
   (b) Obtain $\langle M \rangle$ by the recursion theorem.
   (c) Compute $q(\langle M \rangle)$, and write it to the tape.
   (d) Halt ”

   We let $N = q(\langle M \rangle)$. When $M$ runs, it writes $N = \langle q(\langle M \rangle) \rangle$ in its tape, and when $N$ runs, it writes $\langle M \rangle$ in its tape.

4. **[10 points]** Show that the set of incompressible stings is undecidable.

   **Answer (sketch)**. We solved this one in class. Let $A$ be the set of incompressible strings. The main idea is that if we assume $A$ to be decidable, we can construct an enumerator $E$ that enumerates it. On top of $E$, we construct a Turing Machine $T$, which given a number $n$, returns the first string $w$ that $E$ enumerates of length $n$. If $K(\langle T, n \rangle) = c + \log(n)$, for any constant $c$ we can find $n$ such that $n > c + \log(n)$, which would imply $w$ is compressible.