CS 475: Formal Models of Computation
Homework 5

1. [10 points] Let \((\mathbb{N}, <)\) be the model with universe \(\mathbb{N}\) (natural numbers) and the "less than" relation. Prove that \(\text{Th}(\mathbb{N}, <)\) is decidable.

**Answer.** Proof is practically the same as in Sipser for \(\text{Th}(\mathbb{N}, +)\), pages 227–229. The only change is to construct a DFA that verifies the \(<\) relation, rather than \(+\). Alternatively, use the reduction on Sipser, page 244.

2. [10 points] Prove that the following languages are undecidable (hint: Use Rice’s theorem).

- \(\text{EMPTY}_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}\).
  
  **Answer.** To use Rice’s theorem we need to verify two conditions. For the first one we have that if \(L(M_1) = L(M_2)\), then \(M_1 \in \text{EMPTY}_{TM}\) iff \(M_2 \in \text{EMPTY}_{TM}\). This condition is true by the construction of \(\text{EMPTY}_{TM}\). For the second condition, we need to verify that there is a T.M. not in \(\text{EMPTY}_{TM}\). This is trivially true, since there are machines with non-empty languages. Therefore, by Rice’s theorem, \(\text{EMPTY}_{TM}\) is undecidable.

- \(\text{ALL}_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \Sigma^*\}\).
  
  **Answer.** [From the previous point, substitute \(\text{EMPTY}_{TM}\) with \(\text{ALL}_{TM}\), and “with non-empty languages” with “that reject some string”].

3. [10 points] Describe two different Turing Machines, \(M\), and \(N\), that, when started on any input, \(M\) outputs \(\langle N \rangle\), and \(N\) outputs \(\langle M \rangle\).

**Answer.** To solve this problem, we have to be careful about circular definitions, that is, defining \(M\) in terms of \(N\), and \(N\) in terms of \(M\). Remember that \(q(w)\) is a T.M. that prints \(w\) on its tape, and halts.

\(M = "\) On input \(w:\)

(a) Ignore input.
(b) Obtain \(\langle M \rangle\) by the recursion theorem.
(c) Compute \(q(\langle M \rangle)\), and write it to the tape.
(d) Halt ".

We let \(N = q(\langle M \rangle)\). When \(M\) runs, it writes \(N = \langle q(\langle M \rangle) \rangle\) in its tape, and when \(N\) runs, it writes \(\langle M \rangle\) in its tape.

4. [10 points] Show that the set of incompressible strings is undecidable.

**Answer (sketch).** We solved this one in class. Let \(A\) be the set of incompressible strings. The main idea is that if we assume \(A\) to be decidable, we can construct an enumerator \(E\) that enumerates it. On top of \(E\), we construct a Turing Machine \(T\), which given a number \(n\), returns the first string \(w\) that \(E\) enumerates of length \(n\). If \(K(\langle T, n \rangle) = c + \log(n)\), for any constant \(c\) we can find \(n\) such that \(n > c + \log(n)\), which would imply \(w\) is compressible.