University of Illinois at Urbana-Champaign
Department of Computer Science

Homework Assignment 0

CS 273 Introduction to Theoretical Computer Science
Fall Semester, 2002

Due: Thursday, Sept 5, at the beginnning of class

Some of these are warm-up problems to prepare you for the material of this course.

Please **read** the comments on the web page about **how to do homeworks** before doing this homework. **This is an individual homework.**

1. **(10 pts) Prove that for any positive integer** $n > 1$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

2. **(10 pts) Prove by induction that the sum of the cubes of three consective integers is divisible by** 9**. (Here integers refer to both positive and negative ones.)**

3. **(10 pts) Prove that for any real number** $n \geq 0$ **and integers** $a, b > 0$**,**
   - **(a)** $\lceil \lceil n/a \rceil / b \rceil = \lceil n/ab \rceil$
   - **(b)** $\lfloor \lfloor n/a \rfloor / b \rfloor = \lfloor n/ab \rfloor$

4. **(10 pts) Prove that the number of distinct prime numbers is infinite. (Hint: prove by contradiction)**

5. **(20 pts) Define** $f(n) \ll g(n)$ **to mean that** $f(n)$ **is in** $o(g(n))$ **and** $f(n) \equiv g(n)$ **to mean that** $f(n) = \Theta(g(n))$**. Define** $\lg n = \log_2 n$**.**
   - **(a) Prove that** $n^{1/\lg n} \equiv \sin n + 2 = \Theta(1)$**.**
   - **(b) Order the following functions** $(\lg n)^{\lg n}$**,** $n^{\lg \lg n}$**,** $n^{\lg n}$ **and** $(\lg n)^n$ **by notations** $\ll$ **and** $\equiv$**. Justify your answer.**

6. **(10 pts) Let** R_1 **be a binary relation from set** A **to** B **and** R_2 **be a binary relation from set** B **to** C**. Define** $R_1 \circ R_2 = \{(a, c) | a \in A, c \in C, \exists b \in B$ **such that** $(a, b) \in R_1, (b, c) \in R_2\}$**. Define** $R^2 = R \circ R$**. Prove that a binary relation** R **on set** A **is transitive if and only if** $R \supseteq R^2$**.**

7. **(10 pts) Prove that in a graph** G **with** n **vertices and** $n + 1$ **edges, there is at least one vertex of degree** ≥ 3**.**