Due: Thursday, Sept 5, at the beginnning of class

Some of these are warm-up problems to prepare you for the material of this course.

Please **read** the comments on the web page about **how to do homeworks** before doing this homework. **This is an individual homework.**

1. **(10 pts) Prove that for any positive integer** $n > 1$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

Solution: Prove by induction.
Base Step: When $n = 2$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}} \approx 1.707 > 1.414 \approx \sqrt{2}$. Induction Step: Assume that it's true for $n = k$. When $n = k + 1$,

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} &> \sqrt{k} + \frac{1}{\sqrt{k+1}} \\
&= \frac{\sqrt{k^2+k}+1}{\sqrt{k+1}} \\
&> \frac{\sqrt{k^2}+1}{\sqrt{k+1}} \\
&= \frac{k+1}{\sqrt{k+1}} \\
&= \sqrt{k+1}
\end{aligned}$$

Thus, it's true when $n = k + 1$. This finishes the induction step, which completes the proof. ■

2. **(10 pts) Prove by induction that the sum of the cubes of three consective integers is divisible by** 9**. (Here integers refer to both positive and negative ones.)**

Solution: Let the middle one of the three consective integers be n. Base Step: When $n = 0$,$(-1)^3 + 0^3 + 1^3 = 0$,which is divisible by 9. Induction Step: Assume that it's true for $n = k$. First, we prove the case in which n increases. When $n = k + 1$,

$$\begin{aligned}
k^3 + (k+1)^3 + (k+2)^3 &= (k-1)^3 + k^3 + (k+1)^3 + ((k+2)^3 - (k-1)^3) \\
&= (k-1)^3 + k^3 + (k+1)^3 + 9k^2 + 9k + 9
\end{aligned}$$

Since $9k^2 + 9k + 9$ is divisible by 9, it's true when $n = k + 1$. The case in which n decreases can be proved in the same way. This finishes the induction step, which completes the proof. ■

3. **(10 pts) Prove that for any real number** $n \geq 0$ **and integers** $a, b > 0$**,**

(a) $\lceil \lceil n/a \rceil / b \rceil = \lceil n/ab \rceil$

Solution: When $n = 0$, the proof is trivial. We are concerned with the case when $n > 0$. Since $n > 0, a, b > 0$ and $a, b \in Z$, there exists $k \geq 0, k \in Z$, such that $k < n/a \leq k + 1$, which means $\lceil n/a \rceil = k + 1$. Since $k + 1 \in Z$, there exist unique $q, r \in Z$ and $q \geq 0, 0 \leq r < b$, such that $k + 1 = b \cdot q + r$. Therefore, $\lceil n/a \rceil / b = \frac{k+1}{b} = q + r/b$.

i. When $r = 0$:
$\lceil n/a \rceil / b = q \Rightarrow \lceil \lceil n/a \rceil / b \rceil = q$.

For the right hand of the equation,

$$\begin{aligned} & k < n/a \le k+1 \\ \Rightarrow\ & \frac{k}{b} < n/ab \le \frac{k+1}{b} \\ \Rightarrow\ & q-1 < q + \frac{-1}{b} = \frac{k}{b} < n/ab \le \frac{k+1}{b} = q \\ \Rightarrow\ & q-1 < n/ab \le q \\ \Rightarrow\ & \lceil n/ab \rceil = q \end{aligned}$$

Thus $\lceil \lceil n/a \rceil / b \rceil = \lceil n/ab \rceil$.

ii. When $r \neq 0$:
$\lceil n/a \rceil / b = q + r/b \Rightarrow \lceil \lceil n/a \rceil / b \rceil = q + 1$.
For the right hand of the equation,

$$\begin{aligned} & k < n/a \le k+1 \\ \Rightarrow\ & \frac{k}{b} < n/ab \le \frac{k+1}{b} \\ \Rightarrow\ & q \le q + \frac{r-1}{b} = \frac{k}{b} < n/ab \le \frac{k+1}{b} = q + r/b < q+1 \\ \Rightarrow\ & q < n/ab < q+1 \\ \Rightarrow\ & \lceil n/ab \rceil = q+1 \end{aligned}$$

Thus $\lceil \lceil n/a \rceil / b \rceil = \lceil n/ab \rceil$.

This completes the proof. ■

(b) $\lfloor \lfloor n/a \rfloor / b \rfloor = \lfloor n/ab \rfloor$

Solution: The proof can be done in exactly the same fashion. ■

4. (10 pts) Prove that the number of distinct prime numbers is infinite. (Hint: prove by contradiction)

Solution: We proof by contradiction. Suppose the number of distinct prime numbers is finite, denoted by n. Let the biggest be P_n and the smallest be P_1. We construct a new number $P_{n+1} = \prod_{i=1}^{n} P_i + 1$. First of all, P_{n+1} is a prime number since it can not be divided by any prime number P_i where $1 \le i \le n$. And $P_{n+1} > P_n$, contradicting our assumption that all primes are in the list P_i, $1 \le i \le n$. ■

5. (20 pts) Define $f(n) \ll g(n)$ to mean that $f(n)$ is in $o(g(n))$ and $f(n) \equiv g(n)$ to mean that $f(n) = \Theta(g(n))$. Define $\lg n = \log_2 n$.

(a) Prove that $n^{1/\lg n} \equiv \sin n + 2 = \Theta(1)$.

Solution: $n^{1/\lg n} = n^{\lg 2/\lg n} = n^{\log_n^2} = 2 = \Theta(1)$.
Since $-1 \le \sin n \le 1$, $1 \le \sin n + 2 \le 3 \Rightarrow \sin n + 2 = \Theta(1)$ ■

(b) Order the following functions $(\lg n)^{\lg n}$, $n^{\lg\lg n}$, $n^{\lg n}$ and $(\lg n)^n$ by notations $\ll$ and $\equiv$. Justify your answer.

Solution: $(\lg n)^{\lg n} \equiv n^{\lg\lg n} \ll n^{\lg n} \ll (\lg n)^n$.

i. $(\lg n)^{\lg n} \equiv n^{\lg\lg n}$ can be justified by showing that they are actually equal.

$$(\lg n)^{\lg n} = (2^{\lg\lg n})^{\lg n} = 2^{\lg\lg n \cdot \lg n} = 2^{\lg n \cdot \lg\lg n} = n^{\lg\lg n}$$

ii. Since for all real constants a and b such that $a > 1$,

$$\lim_{n\to\infty} \frac{n^b}{a^n} = 0$$

Substituting $\lg n$ for n and 2^a for a, we have

$$\lim_{n\to\infty} \frac{\lg^b n}{(2^a)^{\lg n}} = \lim_{n\to\infty} \frac{\lg^b n}{n^a} = 0$$

From this limit, we conclude that $\lg^b n \ll n^a$ for any constant $a > 0$. Taking $a = 1, b = 1$,we have $\lg n \ll n$, which gives us $n^{\lg\lg n} \ll n^{\lg n}$.

iii. Taking $a = 1, b = 2$,we have $\lg^2 n \ll n$. Together with the obvious result $2 \ll \lg n$, we have

$$n^{\lg n} = 2^{\lg^2 n} \ll 2^n = 2^{n \lg 2} \ll 2^{n\cdot\lg\lg n} = 2^{\lg\lg n\cdot n} = (\lg n)^n$$

This justifies $n^{\lg n} \ll (\lg n)^n$.

■

6. (10 pts) Let R_1 be a binary relation from set A to B and R_2 be a binary relation from set B to C. Define $R_1 \circ R_2 = \{(a, c)|a \in A, c \in C, \exists b \in B$ such that $(a, b) \in R_1, (b, c) \in R_2\}$. Define $R^2 = R \circ R$. Prove that a binary relation R on set A is transitive if and only if $R \supseteq R^2$.

Solution: (a) We first prove that a binary relation R on set A is transitive if $R \supseteq R^2$.
If $R \supseteq R^2$, whenever $(a, b) \in R$ and $(b, c) \in R$, $(a, c) \in R^2$, and thus $(a, c) \in R$. R is transitive according to definition of transitivity.

(b) We then prove that $R \supseteq R^2$ if a binary relation R on set A is transitive.
If a binary relation R on set A is transitive, whenever $(a, b) \in R$ and $(b, c) \in R$, there exists $(a, c) \in R$. Since for every element $(a, c) \in R^2$, there exists $b \in R$ such that $(a, b) \in R$ and $(b, c) \in R$. Thus for every element $(a, c) \in R^2$, $(a, c) \in R$, which proves $R \supseteq R^2$ by definition.

This completes the proof. ■

7. (10 pts) Prove that in a graph G with n vertices and $n + 1$ edges, there is at least one vertex of degree ≥ 3.

Solution: Prove by contradiction. Suppose for the purpose of contradiction that each and every vertex has a degree of at most 2. The sum of degree of all the n vertices is at most $2n$. However, since there are $n+1$ edges and each and every edge contributes exactly 2 to the sum of degree of G independently. The sum of degree of the graph is exactly $2n + 2$, contradicting the previous result. Therefore, in a graph G with n vertices and $n + 1$ edges, there is at least one vertex of degree ≥ 3. ■