Homework 1 Solutions

1. (10 pts)

   (a) (5 pts) How many ways can ten boys and five girls stand in a *row* with no two girls standing beside each other?

   Solution: Let the 10 boys stand in a line, with an empty position between each pair of them. So there are 9 empty positions in the line. Plus those two positions at the start and the end of the line, totally there are 11 empty positions. It is obvious that if the five girls stand in these empty positions, no two girls will stand beside each other. Given a certain standing order of boys, the number of ways for girls to stand is the number of 5-permutations of a set with 11 elements, this is P(11,5). The boys can also stand in an arbitrarily order. So the number of total ways is $P(11,5) \times P(10,10)$. ■

   (b) (5 pts) How many ways can ten boys and five girls stand in a *circle* with no two girls standing beside each other?

   Solution: If the boys stand in a circle, there are 10 empty positions in the circle. (Note that the start and the end positions of the line merge into one position)So the total ways for girl to stand is P(10,5). In case the boys stand in a circle, we can fix one boy on a fixed position in a circle, the rest 9 boys can stand in an arbitrarily order. So the number if total ways is $P(10,5) \times P(9,9)$. ■

2. (16 pts) There are five male students and five female students register for a 497 seminar course. Unfortunately, the professor has trouble hiring any graders and he himself finds the job unacceptably boring. As a result, he asks the students to exchange homework so that no one grades his or here own submission.

   Solution: Assume there are n students taking the course, define p(n) as the number of ways to exchange homework so that no one grades his or her own submission.

   (a) Solution 1

   Obviously, p(2)=1, p(3)=2. Now we try to get the recursive function of p(n). Let's consider student n, there are two cases:

   1)student n grades the homework of student i, and his homework is graded by student j,where $n \neq i, n \neq j, i \neq j$, if student n is out, we can let j grades i's homework, so the rest n-1 students still grade others' homework. On the other hand, in any feasible way for n-1 students, there are n-1 relation links (such as student i grades j's homework). And student n can break each one of these links to constitute a new feasible way for n students. (by letting i grade n, n grade j) In this case, the total contribution is $(n-1) \times p(n-1)$

   2)student n grades student i's homework, and student i grades student n's homework. There are n-1 candidates to grade homework with student n. The rest n-2 students still grades others' homework. So in this case, the total contribution is $(n-1) \times p(n-2)$. Now we get the recursive function as follows:

   $$p(n) = (n-1) \times p(n-1) + (n-1) \times p(n-2)$$

   Divide n! at both side of the equation, we get

   $$\frac{p(n)}{n!} = \frac{p(n-1)}{(n-1)!} - \frac{p(n-1)}{n \times (n-1)!} + \frac{p(n-2)}{n \times (n-2)!}$$

   so

   $$\frac{P(n)}{n!} - \frac{p(n-1)}{(n-1)!} = \frac{(-1)}{n} \times (\frac{p(n-1)}{(n-1)!} - \frac{p(n-2)}{(n-2)!})$$

   We get the solution as

   $$p(n) = n! \times \sum_{k=2}^{n} \frac{(-1)^{(k)}}{k!}$$

where $n \geq 2$

(b) Solution 2

First, let all the students grades their homework arbitrarily, the number of total ways is P(n,n). Define g(k) as there are exactly k students grade their own homework. So

$$p(n) = P(n,n) - \sum_{k=1}^{n} g(k)$$

Consider the situation that one student is fixed to grade his/her own homework, others can grade homework arbitrarily, there are $C(n,1) \times P(n-1,n-1)$ possibilities. But in this case, we calculate some instances for more than one times. For example, we calculate g(2) C(2,1) times, calculate g(3) C(3,1) times....,calculate g(n) C(n,1) times. We get

$$C(n,1) \times P(n-1,n-1) = \sum_{k=1}^{n} C(n,k) \times g(k)$$

Simiarly, if we fix two students, we have

$$C(n,2) \times P(n-2,n-2) = \sum_{k=2}^{n} C(n,k) \times g(k)$$

And so on...

$$C(n,n-1) \times P(1,1) = \sum_{k=n-1}^{n} C(n,k) \times g(k)$$

$$C(n,n) \times P(0,0) = \sum_{k=n}^{n} C(n,n) \times g(k) = g(n)$$

Now we prove

$$p(n) = P(n,n) - \sum_{k=1}^{n} g(k)$$

$$= P(n,n) - (C(n,1) \times P(n-1,n-1) - C(n,2) \times P(n-2,n-2) + C(n,3) \times P(n-3,n-3) - ...)$$

$$p(n) = n! \times \sum_{k=2}^{n} \frac{(-1)^{(k)}}{k!}$$

Replace $C(n,k) \times P(n-k,n-k)$ (k=1...n) using the equations we get above, we need to prove that

$$\sum_{i=1}^{k} (-1)^{(i)} \times C(k,i) = -1$$

It is trivial if we let x=1, y= -1 in the THEOREM 6 in ROSEN p256.

■

(a) (4 pts) In how many ways can this be accomplished?

Solution: The answer is just p(10), where p(n) is as defined above. ■

(b) (4 pts) In how many ways can at least three students get lucky to grade their own homework?

Solution: Total number- no one grades his/her homework - only one student grades his/her homework - only two students grade their own homework

$$P(10,10) - p(10) - C(10,1)p(9) - C(10,2)p(8)$$

■

(c) (8 pts) Now back to the conditions given in the original problem and let the total number of registered students be n, in how many ways can this be accomplished (without the constraints in b)?

Solution: The answer is p(n). ■

3. (20 pts) Every day a student walks from her home to school, which is located 10 blocks east and 14 blocks north from home. She always takes a shortest walk of 24 blocks. Assume she lives on a prefect grid.

(a) (5 pts) How many different walks are possible?

Solution: A path can be represented by a bit string of length 24. A zero means that the student went right and a one means that the student went up. Note that the student cannot go left or down because the student will not then be making a shortest walk of 24 blocks. Given this path configuration the student can go only 10 blocks to the right $\Rightarrow$ the number of possible paths is: $\binom{10+14}{10} = \binom{24}{10} = 1961256$ ■

(b) (5 pts) Suppose that 4 blocks east and 5 blocks north of her home lives her best friend, whom she meets each day on her way to school. Now how many different ways are possible?

Solution: We have to find the number of ways to go to the best friend first and then from the best friend to school. $\binom{5+4}{4} * \binom{9+6}{6} = \binom{9}{4} * \binom{15}{6} = 630630$ ■

(c) (5 pts) Suppose, in addition, that 3 blocks east and 6 blocks north of ther friend's house there is a park where the two girls stop each day to rest and play. Now how many different walks are there?

Solution: First we will find the number of ways to go to the best friend, then the number of ways to go from the best friend to the park, and finally the number of ways to go from the park to school. This is: $\binom{5+4}{4} * \binom{6+3}{3} * \binom{3+3}{3} = \binom{9}{4} * \binom{9}{3} * \binom{6}{3} = 211680$ ■

(d) (5 pts) Stopping at a park to rest and play, the two students often get to school late. To avoid the temptation of the park, our two students decide never to pass the intersection where the park is. Now how many different walks are there?

Solution: ¿From part b) we know how many ways there are to go from the friend's house to school. ¿From part c) we know how many ways there are to go from the friend's house to school via the park. So subtract the second from the first and we get: $\binom{5+4}{4} * [\binom{9+6}{6} - \binom{6+3}{3} * \binom{3+3}{3}] = \binom{9}{4} * [\binom{15}{6} - \binom{9}{3} * \binom{6}{3}] = 418950$ ■

4. (10 pts) Use *combinatorial* reasoning to prove the identity (in the form given)

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$$

Solution: Imagin that we have a set of distinguishable objects $S = \{a, b, c, s_1, s_2, \ldots, s_{n-3}\}$. Now we want to choose k objects from S with the constraint that at least one of a, b and c must be chosen. Observe that $\binom{n-3}{k}$ is number of ways to choose k objects from S without choosing any one of a, b and c. Therefore the left side of the equation $\binom{n}{k} - \binom{n-3}{k}$ computes exactly the number of ways to choose k objects from S with the constraint that at least one of a, b and c must be chosen. Let's look at the right side of the equation. From another point of view, there are three ways to accomplish the same task:

(a) Object a is chosen.
We then have to choose another $k-1$ objects from the $n-1$ objects left. The number of ways to do this is $\binom{n-1}{k-1}$. Let the set of all the distinct sets of k objects chosen this way be A_1.

(b) Object a is *not* chosen. We throw away object a ($n-1$ objects left)
Object b is chosen. We then go ahead and choose another $k-1$ objects from the $n-2$ objects left. The number of ways to do this is $\binom{n-2}{k-1}$. Let the set of all the distinct sets of k objects chosen this way be A_2.

(c) Both object a and b are *not* chosen. We throw away object a and b ($n-2$ objects left)
Object c is chosen. We then go ahead and choose another $k-1$ objects from the $n-3$ objects left. The number of ways to do this is $\binom{n-3}{k-1}$. Let the set of all the distinct sets of k objects chosen this way be A_3.

Let A be the set of all the distinct sets of k objects chosen from S with the constraint at least one of a, b and c must be chosen. One important observation is that $A_i \cap A_j = \emptyset, 1 \leq i < j \leq 3$ and $A = A_1 \cup A_2 \cup A_3$. Now the right side of the equation computes exactly $|A|$. Both sides computing the same quantity, the equation thus holds. ■

5. (10 pts) Use *combinatorial* reasoning to prove

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

for any $n \geq 0$

Solution: Imagine we have a set of $2n$ objects $S = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$. Now we want to choose n objects from S. The right side of the equation $\binom{2n}{n}$ is just the number of all distinct sets of n objects chosen from S. Let's look at the left side. From another point of view, there are $n+1$ ways in which this can be accomplished. Think about S as $S = A \cup B$ where $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. We can choose nothing from A and choose all n objects from B in $\binom{n}{0}\binom{n}{n}$ ways. Or we can choose one object from A and $n-1$ objects from B in $\binom{n}{1}\binom{n}{n-1}$ ways. Or we can choose two objects from A and $n-2$ objects from B in $\binom{n}{2}\binom{n}{n-2}$ ways. ... Or we can choose n objects from A and no objects from B in $\binom{n}{n}\binom{n}{0}$ ways. Notice that $\binom{n}{k} = \binom{n}{n-k}$. The total number of ways to choose n objects from $S = A \cup B$ is thus $\sum_{k=0}^{n} \binom{n}{k}^2$. Since both sides of the equation compute the same quantity, the equation holds. ■

6. A thousand balloons are to be given to 200 children $C_1, C_2, \ldots, C_{200}$. In how many ways can it be done if

(a) (2 pts) the balloons are identical?

Solution: The children can be represented as the 200 spaces before, between, and after 199 bars. The problem then reduces to how many ways a 1000 balloons can be distributed among these spaces. A sample representation of the problem depicting a balloon as $*$, could be:

$$**\,|\,**||\,*\ldots|\,***$$

It is obvious from the representation that this is equivalent to the number of different permutations of (1000 + 199) objects, where there are 1000 indistinguishable objects of type $*$ and 199 indistinguishable objects of type $|$. By theorem 3, on page 292 of text, the answer is

$$\frac{1199!}{1000!199!}$$

■

(b) (2 pts) the balloons are all different?

Solution: We give out the balloons one by one in order. Since each balloon can be given to any child and there is no restriction no the number of balloons a child can have. The number of ways of doing this is 200^{1000}. To verify the correctness, one can argue that every way of giving out balloon in this fashion is a valid one and every valid configuration can be achieved. Furthermore, since the balloons are given out in order, a single configuration cannot be achieved in two different ways. ■

(c) (3 pts) the balloons are identical and each child must get at least one?

Solution: Once again the children can be represented as the space before and after and the spaces between 199 bars. The task being to distribute 1000 balloons into the different spaces but with each space containing at least 1 balloon. This is really the same as first putting a balloon in each space:

$$*|*|*\dots|*$$

The problem then reduces to how many ways can we put 800 balloons into each space. To see this let b represent the remaining balloons, we can now represent the problem as:

$$*bbb|*bbb|*bbb\dots bb|*bb$$

To stay within the problem constraints, let the first $*$ always be fixed at that position. Also each $|*$ can be thought of as being a representation of a single bar. Since there's only one way to fix the $*$ at the first position, the problem is now equivalent to the number of different permutations of (800 + 199) objects, where there are 800 *indistinguishable* objects of type b and 199 indistinguishable objects of type $|*$. By theorem 3, on page 292 of text, the answer is

$$\frac{999!}{800!199!}$$

■

(d) (3 pts) the balloons are all different and each child must get at least one?

Solution: Let P_i be the property that child i doesn't get any balloons and A_i be the set of distinct ways of giving out balloons with property P_i. If follows that $\overline{P_i}$ is the property that child i has at least one balloon and $\overline{A_i}$ is the corresponding set. The set of all the distinct ways to give out the 1000 balloons such that each child must get at least one can be expressed as $\overline{A_1}\bigcap\overline{A_2}\bigcap\dots\bigcap\overline{A_{200}}$. Let S be the set of all the possible ways to give out the balloons.

$$\begin{aligned}
|\overline{A_1}\bigcap\overline{A_2}\bigcap\dots\bigcap\overline{A_{200}}| &= |S| - \sum_{i=1}^{n}|A_i| + \sum_{1\le i<j\le n}|A_i\bigcap A_j| - \sum_{1\le i<j<k\le n}|A_i\bigcap A_j\bigcap A_k| + \dots + \\
&\quad |A_1\bigcap A_2\bigcap A_3\bigcap\dots A_{200}| \\
&\quad /* \\
&\quad \text{since property } P_i \text{ is symmetric,which means} \\
&\quad |A_1| = |A_2| = \dots = |A_{200}| = 199^{1000} \\
&\quad |A_1\bigcap A_2| = |A_1\bigcap A_3| = \dots = |A_{199}\bigcap A_{200}| = 198^{1000} \\
&\quad \dots \\
&\quad |A_1\bigcap A_2\bigcap\dots\bigcap A_{199}| = \dots = |A_2\bigcap A_3\bigcap\dots\bigcap A_{200}| = 1^{1000} \\
&\quad */ \\
&= 200^{1000} - \binom{200}{1}199^{1000} + \binom{200}{2}198^{1000} - \dots + \binom{200}{198}2^{1000} \\
&\quad - \binom{200}{199}1^{1000} + \binom{200}{200}\cdot 0
\end{aligned}$$

■

7. (24 pts) Consider strings of 7 digits (e.g., 0000000, 1345692)

(a) (4 pts) How many are there?

Solution: For each digit, there are 10 choices $(0, 1, 2, \ldots, 9)$. Thus, there are 10^7 strings of 7 digits. ■

(b) (5 pts How many contain at least one 2?

Solution: Let P_0 be the property that a string contains no 2's and A_0 be the set of strings with property P_0. Then, it's easy to see that $\overline{A_0}$ is exactly the set of strings each of which contains at least one 2. For each string in A_0, there are 9 choices for each digit. Therefore $|A_0| = 9^7$. $|\overline{A_0}| = 10^7 - |A_0| = 10^7 - 9^7 = 5217031$. ■

(c) (7 pts) Suppose there are exactly one 2,two 5's and four 7's in any string. How many such strings are there?

Solution: This is the problem of counting permutations of mutiple sets of indistinguishable elements. Observe that for any one permutation of the 7 digits (one 2,two 5's and four 7's), changing the positions of the indistinguishable elements of a set does not change the ordered arrangement of the entire permutation. In general, given k sets $S_1, S_2, \ldots, S_k$, each of which contains $n_i, 1 \leq i \leq k$ indistinguishable elements, where $\sum_{i=1}^{k} n_i = n$, the total number of distinct permutations of these n elements is $\frac{n!}{n_1! \cdot n_2! \cdots n_k!}$. Thus, the answer is $\frac{7!}{1! \cdot 2! \cdot 4!} = 105$. ■

(d) (8 pts) Suppose there are at most one 2's, three 5's and five 7's in any string. How many such strings are there?

Solution: Let P_1 be the property that a string has at least two 2's, P_2 be the property that a string has at least four 5's and P_3 be the property that a string has at least six 7's. And let A_1, A_2 and A_3 be the corresponding sets of strings. $|A_1| = 10^7 - 9^7 - \binom{7}{1}9^6 = 1496944$. $|A_2| = \binom{7}{4}9^3 + \binom{7}{5}9^2 + \binom{7}{6}9 + 1 = 27280$. $|A_3| = \binom{7}{6}9 + 1 = 64$. $|A_1 \cap \mathcal{A}_2| = \frac{7!}{3!4!} + \frac{7!}{2!5!} + 8 \cdot \frac{7!}{2!4!1!} = 896$ since given that we already have two 2's and four 5's, choosing a seventh digit gives us a configuration of mutiple sets of indistinguishable elements, which can be solved by the formula in (b). Since we are only dealing with strings of length 7, $|A_1 \cap A_3| = |A_2 \cap A_3| = |A_1 \cap A_2 \cap A_3| = 0$. The set of the strings that has at most one 2's, three 5's and five 7's is $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$. By the principle of inclusion and exclusion, $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |\overline{A_1 \cup A_2 \cup A_3}| = 10^7 - |A_1 \cup A_2 \cup A_3| = 10^7 - (\sum_{i=1}^{3} |A_i| - \sum_{i<j} |A_i \cap A_j| + |A_1 \cap A_2 \cap A_3|) = 10^7 - 1496944 - 27280 - 64 + 896 = 8476608$. ■