University of Illinois at Urbana-Champaign
Department of Computer Science

Homework Assignment 2

CS 273 Introduction to Theoretical Computer Science
Fall Semester, 2002

Due: Thursday, Oct. 3, at the beginnning of class

Please **read** the comments on the web page about **how to do homeworks** before doing this homework. **This is a group homework. In all the problems, you must explicitly state your reasoning to get credit.**

1. **(20 pts) Suppose there are 40 lottery balls, each numbered 1 through 40. The special lottery ball selection machine selects a ball at random (without replacement), each time it is asked to provide a ball. Initially, all 40 balls are in the machine. Suppose that we perform a sequence of trials in which a ball is drawn each time, and we stop whenever the lucky 7 ball is drawn.**

   (a) **(6 pts) What is the expected number of balls drawn?**

   (b) **(6 pts) What is the expected value of the smallest-numbered ball?**

   (c) **(8 pts) What is the expected value of the largest-numbered ball?**

   Give exact answers and show that they are correct. Answers based on intuition could easily be wrong.

2. **(12 pts) A ship arrives at a port, and 220 sailors on board go ashore for revelry. Later at night, the 220 sailors return to the ship and, in their state of inebriation, each chooses a random cabin to sleep in.**

   (a) **(3 pts) What is the expected number of sailors sleeping in their own cabins, assuming that there is exactly one sailor per cabin?**

   (b) **(3 pts) Suppose that the sailors are so plastered that each passes out in a cabin regardless of whether there are already other sailors in it. What is the expected number of sailors sleeping in their own cabin in this case?**

   (c) **(3pts) In the latter case, what is the expected number of sailors sleeping alone in their own cabin?**

   (d) **(3 pts) Again, assuming the latter case, what is the expected number of sailors sleeping alone in a cabin, regardless of whether it is their own?**

3. **(8 pts) Prove that among any $n+1$ distinct positive integers not exceeding $2n$, there exist two integers that are relatively prime (they share no common positive factors except 1).**

4. **(20 pts)**

   An unbiased coin has a probability of $\frac{1}{2}$ to be HEAD. However, imagine you only have a biased coin whose probability of HEAD is an unknown value p.

   (a) **(6 pts) Is it possible to generate an unbiased random bit with this biased coin? Thus, you have to be able to produce a ONE or ZERO, each with probability 1/2. If possible, explain how.**

(b) (6 pts) Suppose someone suggests the following method. Toss the coin twice: if the outcome is TH, output the bit ZERO; if HT, output the bit ONE; anything else, NO OUTPUT and you must repeat the experiment until you can output a bit.

Does this method work? Explain why you think it works, or doesn't work.

(c) (8 pts) Regardless of whether the method in (b) works correctly, what is the expected number of coin tosses before a bit can be generated?

5. (10 pts) There are three coins. One has both sides painted black and one has both sides painted red. The other one has one side painted black and the other side painted red. Now imagine you take one coin out of a pocket containing only the three coins, and place it down on a table. Both the particular coin and the side that is face up are randomly chosen. Looking straight down on the coin, you see that the face that is up is red. Given this observation, what is the probability that this coin is black on the other side? You must justify your answer.

6. (10 pts) Suppose you bought a cheap computer by mail order, and have had a series of problems. Each time you try calling the service department, and one of the following can happen:

(a) The line is busy (event E_1);

(b) You get disconnected (event E_2);

(c) They tell you it's the wrong number (event E_3);

(d) The service representative does not speak English (event E_4);

(e) You actually speak to someone who is able to help (event E_5).

Assume $P[E_i] = p_i$, and that all of the events above are independent.

(a) (5 pts) Suppose that you have to make 12 calls to the service department. What is the probability that the line is busy seven times, you get disconnected once, the service representative does not speak English two times, and twice you are told it is the wrong number? (Of course you never speak to someone who is able to help.)

(b) (5 pts) What is the probability that a busy signal is obtained at least 10 times of the 12 calling attempts?

7. (10 pts) Consider functions mapping from $\{a_1, a_2, a_3, a_4, a_5\}$ to $\{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$. Suppose that a sample space consists of the set functions. Suppose a function is chosen at random.

(a) (2 pts) What is the probability that the function is injective (one-to-one)?

(b) (4 pts) What is the probability that the function is surjective (onto)?

(c) (4 pts) What is the probability that the function is bijective (one-to-one and onto)?

8. (10 pts) Suppose that you have r indistinguishable balls, and distinguishable n boxes. Each ball is placed into a box at random.

(a) (5 pts) Derive a formula for the probability that the first box contains at least k balls, for $0 < k \leq r$.

(b) (5 pts) Derive a formula for the probability that at most k boxes are empty, for some $0 < k \leq n$.