Due: Thursday, Oct. 3, at the beginnning of class

Please **read** the comments on the web page about **how to do homeworks** before doing this homework. **This is a group homework. In all the problems, you must explicitly state your reasoning to get credit.**

1. **(20 pts) Suppose there are 40 lottery balls, each numbered 1 through 40. The special lottery ball selection machine selects a ball at random (without replacement), each time it is asked to provide a ball. Initially, all 40 balls are in the machine. Suppose that we perform a sequence of trials in which a ball is drawn each time, and we stop whenever the lucky 7 ball is drawn.**

   (a) **(6 pts) What is the expected number of balls drawn?**

   Solution: Define random variable X to be the number of balls drawn until the ball numbered 7 is drawn. Then the expected number of balls drawn is $E[X] = \sum_{i=1}^{40}(Pr[X = i] \times i)$. Since the probability of the ball numbered 7 to be drawn as the ith ball is the same for all $1 \leq i \leq 40$, $Pr[X = i] = 1/40$. Therefore we have

   $$E[X] = \sum_{i=1}^{40}(Pr[X = i] \times i) = \frac{1}{40} \times \sum_{i=1}^{40} i = \frac{1}{40} \times \frac{40}{2} \times 41 = 20.5$$

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   (b) **(6 pts) What is the expected value of the smallest-numbered ball?**

   Solution: Define random variable X to be the number of the smallest ball drawn. $Pr[X = 1]$ means ball numbered 1 is drawn before ball 7. The probability for that event is 1/2, since there are only two relative order of the two element , namely 1 before 7 and 1 after 7. Thus $Pr[X = 1] = 1/2$. For $Pr[X = 2]$,it's sufficient to consider the relative order of the three balls 1,2 and 7. All the other balls are irrelevant. Since out of the 3! ways of permutation of the three balls, only the order (271) makes the event $X = 2$ happen, $Pr[X = 2] = 1/3!$. For $Pr[X = 2]$,it's sufficient to consider the relative order of the four balls 1,2,3and 7. Since out of the 4! ways of permutation of the four balls, ball 2 has to appear before 7 but ball 1 and ball 2 have to appear after 7 to make the event happen. Hence $Pr[X = 3] = \frac{1}{4!} \cdot 2!$. In general, for $Pr[X = i] = \frac{1}{(i+1)!} \cdot (i-1)!, 1 \leq i \leq 6$. We have

   $$E[X] = \sum_{i=1}^{6} \frac{(i-1)!}{(i+1)!} \cdot i + \frac{6!}{7!} \cdot 7 = 1 + 1/2 + 1/3 + \ldots + 1/7 = 1089/420 = 2.6$$

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   (c) **(8 pts) What is the expected value of the largest-numbered ball?**

   Solution: The reasoning is quite similar with part (b). Define random variable X to be the number of the largest-numbered ball drawn. Now consider $Pr[X = 40]$, it is just case in part when we want the probability of the smallest-numbered ball being ball 1. The probability is 1/2. Following the same reasoning, $Pr[X = 39] = \frac{1}{3!}$, $Pr[X = 38] = \frac{2!}{4!}$ and so on. We get

   $$\begin{aligned} E[X] &= \sum_{i=40}^{8} \frac{(40-i)!}{(42-i)!} \cdot i + \frac{33!}{34!} \cdot 7 \\ &= (1 - \frac{1}{2}) \cdot 40 + (\frac{1}{2} - \frac{1}{3}) \cdot 39 + \ldots + (\frac{1}{33} - \frac{1}{34}) \cdot 8 + \frac{7}{34} \\ &= 40 + (39 - 40) \cdot \frac{1}{2} + (38 - 39) \cdot \frac{1}{3} + \ldots + (7 - 8) \cdot \frac{1}{34} \\ &= 41 - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{34} \\ &= 41 - H_{34} \approx 41 - 4.1476 \approx 36.9 \approx 37 \end{aligned}$$

$\blacksquare$

2. **(12 pts) A ship arrives at a port, and 220 sailors on board go ashore for revelry. Later at night, the 220 sailors return to the ship and, in their state of inebriation, each chooses a random cabin to sleep in.**

   (a) **(3 pts) What is the expected number of sailors sleeping in their own cabins, assuming that there is exactly one sailor per cabin?**

   Solution: Define random variable X be the number of sailors sleeping in their own cabins. Define indicator random variable X_i such that

$$X_i = \begin{cases} 1 & \text{if sailor } i \text{ sleeps in his own cabin} \\ 0 & \text{otherwise} \end{cases}$$

   Since there is exactly one sailor per cabin, each configuration is a random permutation of the 220 sailors. It follows that the probability that a sailor ends up in his own cabin is $\frac{219!}{220!} = \frac{1}{220}$.

$$E[X] = E[\sum_{i=1}^{220} X_i] = \sum_{i=1}^{220} E[X_i] = \sum_{i=1}^{220} Pr[X_i = 1] = \sum_{i=1}^{220} \frac{1}{220} = 1$$

$\blacksquare$

   (b) **(3 pts) Suppose that the sailors are so plastered that each passes out in a cabin regardless of whether there are already other sailors in it. What is the expected number of sailors sleeping in their own cabin in this case?**

   Solution: We define the random variable X and indicator random variable exactly the same way as in part (a). Observe that since each sailor uniformly and randomly picks a cabin to sleep when he returns to the ship, without caring if there is already someone in the cabin, the probability that the one he picks is his own cabin is $\frac{1}{220}$. The other analysis is exactly the same as in part (a). Hence the answer is again 1. $\blacksquare$

   (c) **(3pts) In the latter case, what is the expected number of sailors sleeping alone in their own cabin?**

   Solution: Define random variable X be the number of sailors sleeping alone in their own cabins. Define indicator random variable X_i such that

$$X_i = \begin{cases} 1 & \text{if sailor } i \text{ sleeps alone in his own cabin} \\ 0 & \text{otherwise} \end{cases}$$

   Now let's consider $Pr[X_i = 1]$. The size of the sample space in the latter case is 220^{220} since each sailor has 220 choices. Fixing sailor i *alone* in his own cabin, there are 219^{219} ways for the other sailor to sleep. $Pr[X_i = 1]$ is thus $\frac{219^{219}}{220^{220}}$.

$$E[X] = E[\sum_{i=1}^{220} X_i] = \sum_{i=1}^{220} E[X_i] = \sum_{i=1}^{220} Pr[X_i = 1] = \sum_{i=1}^{220} \frac{219^{219}}{220^{220}} = (\frac{219}{220})^{219}$$

$\blacksquare$

   (d) **(3 pts) Again, assuming the latter case, what is the expected number of sailors sleeping alone in a cabin, regardless of whether it is their own?**

   Solution: Define random variable X be the number of sailors sleeping alone in their own cabins. Define indicator random variable X_i such that

$$X_i = \begin{cases} 1 & \text{if sailor } i \text{ sleeps alone in a cabin} \\ 0 & \text{otherwise} \end{cases}$$

Now let's consider $Pr[X_i = 1]$. The size of the sample space is again 220^{220} since each sailor has 220 choices. Sailor i has 220 choices to sleep alone in one of the 220 cabins. After sailor i has chosen his private cabin, there are 219^{219} ways for the other sailor to sleep. $Pr[X_i = 1]$ is thus $\binom{220}{1} \cdot \frac{219^{219}}{220^{220}}$.

$$E[X] = E[\sum_{i=1}^{220} X_i] = \sum_{i=1}^{220} E[X_i] = \sum_{i=1}^{220} Pr[X_i = 1] = \sum_{i=1}^{220} 220 \cdot \frac{219^{219}}{220^{220}} = (\frac{219}{220})^{219} \cdot 220 \approx 82$$

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3. **(8 pts) Prove that among any $n+1$ distinct positive integers not exceeding $2n$, there exist two integers that are relatively prime (they share no common positive factors except 1).**

   Solution: We have integers from 1 all the way up to 2n. Now we pair two consecutive integers together, ex. (1, 2) (3, 4)...(2n-1 2n). Thus we end up with n pairs. But we pick n+1 distinctive positive integers, so two integers come from the same pair, which means that two integers must be consecutive; and since consecutive integers are relatively prime, these will be relatively prime. ■

4. **(20 pts)**

   An unbiased coin has a probability of $\frac{1}{2}$ to be HEAD. However, imagine you only have a biased coin whose probability of HEAD is an unknown value p.

   (a) **(6 pts) Is it possible to generate an unbiased random bit with this biased coin? Thus, you have to be able to produce a ONE or ZERO, each with probability 1/2. If possible, explain how.**

   Solution: Yes, look at the method explained in part b) ■

   (b) **(6 pts) Suppose someone suggests the following method. Toss the coin twice: if the outcome is TH, output the bit ZERO; if HT, output the bit ONE; anything else, NO OUTPUT and you must repeat the experiment until you can output a bit.**
   Does this method work? Explain why you think it works, or doesn't work.

   Solution: Yes this method works because if p is the probability to get a head then consider the experiment where you toss the coin twice. The possible outcomes and their corresponding probabilities are: HH pp HT p(1 - p) TH (1 - p)p TT (1 - p)(1 - p)

   From here we see that the probability of HT and TH are the same, namely p(1- p). Hence the method works. ■

   (c) **(8 pts) Regardless of whether the method in (b) works correctly, what is the expected number of coin tosses before a bit can be generated?**

   Solution: The number of experiments to conduct to get a bit is just one over the probability of getting a bit: 1/[2p(1 - p)]. And since each experiment consists of two tosses then the total number of tosses to get a bit is 2 * 1/[2p(1 - p)] = 1/[p(1 - p)] ■

5. **(10 pts) There are three coins. One has both sides painted black and one has both sides painted red. The other one has one side painted black and the other side painted red. Now imagine you take one coin out of a pocket containing only the three coins, and place it down on a table. Both the particular coin and the side that is face up are randomly chosen. Looking straight down on the coin, you see that the face that is up is red. Given this observation, what is the probability that this coin is black on the other side? You must justify your answer.**

   Solution: Let E be the event of picking a coin with the face down being black. Let F be the event of picking a coin with the face up being red. F is the same as "the event of picking the red/red coin" or "picking the red/black coin with the face up being red". Thus the intersection, $E \bigcap F$ will be the event of picking the red/black coin with the face up being red. $(E|F)$ is the event of picking a coin with the face down being black given that the face up is red.

Conditional Probability is given by:

$$P(E|F) = \frac{P(E \bigcap F)}{P(F)}$$

$$P(F) = \frac{1}{3} + (\frac{1}{3} * \frac{1}{2}) = \frac{1}{2}$$

$$P(E \bigcap F) = \frac{1}{3} * \frac{1}{2} = \frac{1}{6}$$

$$P(E|F) = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

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6. (10 pts) Suppose you bought a cheap computer by mail order, and have had a series of problems. Each time you try calling the service department, and one of the following can happen:

(a) The line is busy (event E_1);

(b) You get disconnected (event E_2);

(c) They tell you it's the wrong number (event E_3);

(d) The service representative does not speak English (event E_4);

(e) You actually speak to someone who is able to help (event E_5).

Assume $P[E_i] = p_i$, and that all of the events above are independent.

(a) (5 pts) Suppose that you have to make 12 calls to the service department. What is the probability that the line is busy seven times, you get disconnected once, the service representative does not speak English two times, and twice you are told it is the wrong number? (Of course you never speak to someone who is able to help.)

Solution: There are C(12,7) ways to have the line busy seven times, afterwhich there'd be C(5,1) ways to get disconnected once, C(4,2) ways to twice get a service representative that doesn't speak english and C(2,2) ways to get the wrong number twice. If E is the event that all of the above happen then:

$$P(E) = \binom{12}{7}\binom{5}{1}\binom{4}{2}(P(E_1)^7 P(E_2) P(E_3)^2 P(E_4)^2)$$

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(b) (5 pts) What is the probability that a busy signal is obtained at least 10 times of the 12 calling attempts?

Solution: Getting a busy signal at least 10 times is equivalent to getting a busy signal 10 times, or getting a busy signal 11 times or getting a busy signal 12 times.

$$P(E) = \binom{12}{10}(P(E_1)^{10}(1 - P(E_1))^2) + \binom{12}{11}(P(E_1)^{11}(1 - P(E_1))) + P(E_1)^{12}$$

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7. **(10 pts) Consider functions mapping from $\{a_1, a_2, a_3, a_4, a_5\}$ to $\{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$. Suppose that a sample space consists of the set functions. Suppose a function is chosen at random.**

   (a) **(2 pts) What is the probability that the function is injective (one-to-one)?**

   Solution: The total number of functions mapping from

   $$A = \{a_1, a_2, a_3, a_4, a_5\}$$

   to

   $$B = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$$

   is $7^5 = 16,807$

   The number of injective function is P(7,5)=2,520

   The probability that the function is injective is $\frac{2520}{16807} = 15\%$. ■

   (b) **(4 pts) What is the probability that the function is surjective (onto)?**

   Solution: Since the domain is smaller than the range, it's impossible for the function to be onto. The probability is thus 0. ■

   (c) **(4 pts) What is the probability that the function is bijective (one-to-one and onto)?**

   Solution: It follows that the probability that the function is bijective (one-to-one and onto) is 0. ■

8. **(10 pts) Suppose that you have r indistinguishable balls, and distinguishable n boxes. Each ball is placed into a box at random.**

   (a) **(5 pts) Derive a formula for the probability that the first box contains at least k balls, for $0 < k \leq r$.**

   Solution: First we calculate the total number of ways to put r balls into n boxes. The problem is equivalent to calculate the total number of ways to put $r + n$ balls into n boxes with at least one ball in each box. The problem can be solved as putting $n - 1$ partition in the $r + n - 1$ slots. So the total number of ways is $C(r + n - 1, n - 1)$.

   Now we calculate the total number of ways with first box containing at least k balls. This problem is equivalent to calculate the total number of ways to put $r - k$ balls into n boxes. We get the number of ways is $C(r + n - k - 1, n - 1)$.

   So the probability is $\frac{C(r+n-k-1,n-1)}{C(r+n-1,n-1)}$.

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   (b) **(5 pts) Derive a formula for the probability that at most k boxes are empty, for some $0 < k \leq n$.**

   Solution: We first calculate the number of ways to put balls in boxes with exactly l boxes being empty. As the boxes are distinguishable, there are $C(n, l)$ ways to select the empty boxes. Now we put the r balls into $n - l$ boxes with at least one ball in each box. We know that is $C(r - 1, n - l)$.

   So the probability is $\frac{\sum_{i=0}^{k} C(n,i) \times C(r-1,n-i)}{C(r+n-1,n-1)}$. ■