Due: Tuesday, Nov. 12, at the beginnning of class

This is a group homework. Please staple your homework in four pairs. 1 and 2, 3 and 4, 5 and 6, 7 and 8. Hand in each of them to the corresponding stack in class.

1. **(10 pts) Let G_n be the graph whose vertices are the permutations of $\{1, \ldots, n\}$ with two permutations $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ adjacent if they differ by interchanging a pair of adjacent entries (G_3 shown below). Prove that G_n is connected.**

   Solution: Given a sequence of N numbers there are (N-1) pairs of "adjacent entries" within the sequence. For example sequence "312", has 2 pairs of adjacent entries: "31" and "12". Thus each vertex in G_n has (N-1) adjacent vertices, since it's adjacent vertices can be found by interchanging the entries within ONE of it's pairs of adjacent entries (definition of G_n). For example a node labelled "312" in G_3 has adjacent vertices "132" and "321". ¿From this we note that no single vertice of Gn can be isolated since it has (N-1) adjacent vertices. Also no component of G_n can be isolated since this would mean that the component has N vertices, and each of those vertices has the other (N-1) vertices in the component as its adjacent vertices. This is impossible, after selecting a vertex V_1, and connecting it to it's N-1 adjacent vertices to form the component, each of V_1's adjacent vertices will have their own different adjacent vertices (in fact the only one that will be adjacent to it within the component is V_1). This follows from the definition of G_n which allows for only one adjacent entry pair to be interchanged. Since no component can be isolated G_n is connected. ■

2. **(10 pts) Use induction on the number of edges to prove that absence of odd cycles is a sufficient condition for a graph to be bipartite.**

   Solution: First we show that any graph with only even cycles can be 2colored

   (a) First note that every graph consists of paths and cycles (could be 0 length).

   (b) Any path can be 2-colored. Inductive Proof: Start by a vertex, color by the first color. Pick all the neighbours, use the other color. Repeat this procedure, and since there is no cycle, no color conflict will ever occur

   (c) Any cycle of even length can be 2-colored. Proof by induction: Two vertices connected by two edges, hence a cycle of length 2, is trivially true. Assume that a 2k-vertex cycle is 2-colorable. To obtain a (2k+2)-vertex cycle add 2 new vertices between, say, a blue and a red vertex this way: b-r_n-b_n-r, where b_n and r_n denote the newly added vertices

   (d) Any 2-colored cycle and/or path can be connected to each other without conflict, as long as no odd-length cycle is produced in the process. Connecting paths and cycles is trivial. Just flip all the colors of one component if necessary. So is connecting two cycles with a single common vertex. The trick is in connecting cycle C_1 and C_2 by two (or more) common vertices. Suppose two cycles are connected by v_1 and v_2. C_1 contains the path P_1, where $P_1 = v_1, \ldots, v_2$. Similarly C_2 contains the path P_2, where $P_2 = v_1, \ldots, v_2$. If the paths are both odd, or both even, the connection points agree on their color (we may need to flip the colors of one component again). But if one of the paths is even, and the other is odd, then there will be a color conflict. However, there will be an odd-length cycle (namely P_1-P_2) and this possibility contradicts the assumption of no odd cycles, thus it is ruled out. Hence, a graph can be 2-colored if the ! cy! cles are of even length

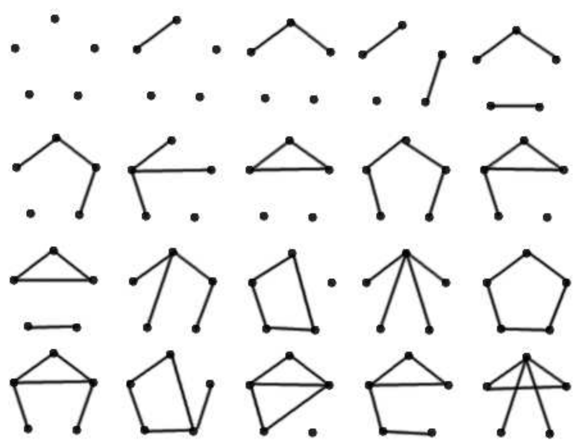
   Since the lack of odd cycles implies only even cycles and also any 2colored graph is bipartite (just put the 2 different colors in seperate sets), the above proof is sufficient. ■

3. **(10 pts) Given that every walk of length $l-1$ contains a path from its first vertex to its last, prove that every walk of lenth l also satisfies this.**

   Solution: (proof by induction)
   Basis: When $l = 1$ then we have a walk of length zero so there is nothing to prove and the claim holds vacuously.
   Induction Hypothesis: Suppose that for any walk of length less then or equal to $l - 1$ there is a path from the first to the last vertex.
   Induction Step: Given a walk of length l show that their is a path from its first to its last vertex. To do this just split the path into two parts one going from the first vertex (call it A) up until the one before the last one (call it X), and one from X to the last vertex (call it B). From the induction hypothesis we know that there is a path from A to X, and from X to B, since both of these walks have a legth less then or eqaual to $l - 1$. Hence a path exist from A to B. ■

4. **(10 pts) For a vertex set of size n, there are $2^{\binom{n}{2}}$ simple graphs. However, these graphs can be divided into disjoint isomorphism classes. For example, there are only 11 isomorphism classes for vertex set of size 4. Dertermine the number of isomorphism classes for vertex set of size 5.**

   Solution: Look at the figure bellow, note that we only need to figure out how many graphs are there having edges from 0-5, since the graphs with edges from 6-10 are just compliments of the graphs with 4-0 edges respectively (so they will have the same behavior). Total number of classes: $1 + 1 + 2 + 4 + 6 + 6 + 6 + 4 + 2 + 1 + 1 = 34$



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5. **(10 pts) Determine which pairs of graphs below are isomorphic, presenting the proof by testing the smallest possible number of pairs.**

Solution: If two graphs are isomorphic, they must have the same number of triangles. The number of triangles for the first,second the fifth graphs are 7, for the third the forth graph is 6.

The third graph and the forth graph are isomorphic. Map the apex of third graph to the central vertex of forth graph.

The first graph and the fifth graph are isomorphic. Label the apex of each graph as 1, and 2,3,4,5,6,7 in the order of clockwise. Map vertex 2,3,4,5,6,7 of graph 1 to vertex 5,2,6,3,7,4 of graph 5. ■

6. **(10 pts) Let G be a tree with n vertices, k leaves and maximum degree k. Prove that G is the union of k paths with a common endpoint. (The union, G, of two graphs, G_1 and G_2, is defined as $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.)**

Solution: We first prove that if there is a tree with k leaves and maximum degree k, there is only one vertex P in the tree whose degree is k, and the maximum degree for all other vertices is 2.

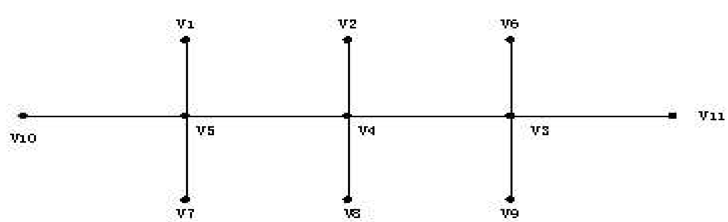
Assume there is another vertex Q whose degree is larger than 2, say n ($n \geq 3$). Starting from P, there are k edges, each of them will finally end with a leaf, and these leaves can not be same, otherwise there will be a loop. Starting from Q, there are n edges, each of them will also finally end with a leaf. There is at most one leaf who is also a leaf starting from P. Otherwise there will be a loop between P and Q. Now we have at least k+n-1 leaves, $n \geq 3$, the total number of leaves will larger than k. Contradiction!

It is trivial that each vertex except P is either a leaf or a node in a path. So the tree is the union of k paths with the common endpoint P. ■

7. **(10 pts) Draw and label a tree whose Prüfer sequence is**

$$5, 4, 3, 5, 4, 3, 5, 4, 3$$

Solution: Let $\delta = 5, 4, 3, 5, 4, 3, 5, 4, 3$ be our initial sequence to which we wish to assign a particular labeled tree. Since there are nine terms in the sequence, our labels will come from the set $S = \{v_1, v_2, \ldots, v_{11}\}$. After drawing the eleven vertices, we look in the set $S = S_0$ to find the smallest subscript that does not appear in the sequence $\delta = \delta_0$. Subscript 1 (V_1) is the one, and so we place an edge between vertices v_1 and v_5, the first subscript in the sequence. We now remove the first term from the sequence and label v_5 from the set, forming a new sequence $\delta_1 = 4, 3, 5, 4, 3, 5, 4, 3$ and a new set $S_1 = \{v_2, v_3, \ldots v_{11}\}$. And so on. The tree is shown as below.



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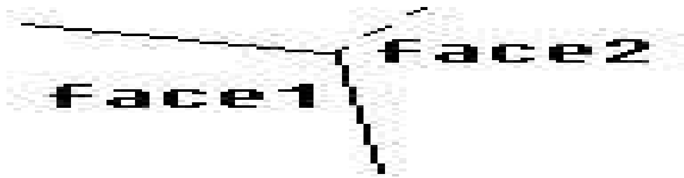
8. **(10 pts) Draw a planar graph with 5 faces and every two faces are bordered by exactly one common edge. Or prove that such a graph does not exist.**

Solution: We will prove that such a graph does not exist. Assume such a graph G exist.

First, we show that the degree of each vertex in G must be no less than 3. See Fig (8-1), if the degree of a certain vertex is 2, then there must be two faces that share at least two common edges.



Let e be the total number of edges in G, v be the total number of vertex in G. Because each edge contributes two degrees, there are exactly 2e degrees in G. So we have $\frac{2e}{v} \geq 3$, that is $v \leq \frac{2e}{3}$

Suppose the number of edges of each face is e_i where $i = 1, 2, 3, 4, 5$. It is trivial that $e_i \geq 4$, because each face have exactly 4 edges to share with other faces. There are totally $C(5, 2) = 10$ common edges. So we have $e \geq \sum e_i - 10 \geq 10$.

According to the Euler's formula,

$$5 = e - v + 2 \geq e - \frac{2e}{3} + 2 \geq \frac{10}{3} + 2$$

It can not be true, so such a graph does not exist.

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