solutionSolution

CS 273: Intro. to Theory, Spring 2001

Midterm Solutions

March 6, 2001

1. Basic Counting (20 points total)

   A wedding photographer is taking pictures of the bride and groom and four other couples. He wants to arrange all ten people in a row.

   (a) (5 points) How many total ways are there to arrange all ten people in a row?

   This is the number of permutations of ten people, which is 10!.

   (b) (5 points) How many arrangements are there so that no two people of the same sex are next to each other?

   This can be divided into two cases. If we number the seats 1 to 10 from left to right, then the men could occupy the odd numbered seats and the women the even numbered seats, or vice versa.

   In the first case, there are 5! ways to arrange the men and 5! ways to arrange the women. So the number of ways to have men in odd seats and women in even seats is $5! \cdot 5!$. There are the same number of ways in the second case. So the total number of ways is $2 \cdot 5! \cdot 5!$.

   (c) (5 points) How many arrangements are there so that every couple is sitting together?

   If every couple should be sitting together, then the photographer can divide the ten seats into pairs, and assign couples to each pair of seats. Since there are five couples, there are 5! ways to assign them to seats. And for each pair, either the man is to the right of the woman, or vice versa. So the correct answer is $2^5 \cdot 5!$.

   (d) (5 points) After the wedding pictures have been taken, the five couples sit down to dinner at a round table (i.e. only relative positions matter). Now how many arrangements are there so that every couple is sitting together?

   This is almost the same question as part (c). First divide the ten positions into pairs. Then decide which couple will sit in which pair of seats. Finally, among each couple, decide who sits on the right.

2. The Pigeonhole Principle (12 points total)

   (a) (5 points) A computer network consists of n computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

   This is a generalization of a homework problem. Suppose the ith computer is connected to x_i other computers. Since there are n computers, each x_i is somewhere in the range of 1 to $n-1$. I.e. there are $n-1$ possible values for each x_i. And since there are n values $\{x_1, x_2, \ldots, x_n\}$, at least two of them are the same.

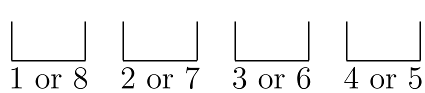
   In terms of pigeonholes, there are $n-1$ pigeonholes corresponding to the possible number of connections. And there are n computers, so one of the pigeonholes has at least two computers.

   (b) (7 points) Show that if five distinct integers are selected from the first eight positive integers $\{1, 2, \ldots, 8\}$, there must be at least one pair of these selected integers with a sum of 9.

   There are four possible ways of having two integers that sum to 9:

$$\{\ 1+8\ ,\ 2+7\ ,\ 3+6\ ,\ 4+5\ \}.$$

   Suppose we have four pigeonholes labelled like this:



If we place five distinct integers into these four pigeonholes, then at least one pigeonhole will have two integers. And their sum will be 9.

3. Probability (25 points total)

When a fair six-sided die is rolled, it is equally likely that any of the numbers $\{1, 2, 3, 4, 5, 6\}$ will appear. Suppose an experiment involves rolling five such identical dice.

(a) (5 points) What is the probability that exactly one 3 appears?

For these questions (and most questions on probability), it's easier to assume that each die is distinct.

First count the number of ways to have a favorable outcome. There are 5 ways to choose which die will have the 3. The other four dice will have numbers other than 3, and there are 5^4 ways for that to happen. So the total number of favorable outcomes is $5 \cdot 5^4 = 5^5$.

And the total number of possible outcomes is 6^5, so the probability is $\frac{5^5}{6^5} = \left(\frac{5}{6}\right)^5$.

(b) (5 points) What is the probability that no 3's appear?

The probability that any particular die is not a 3 is $\frac{5}{6}$, so the probability that all five dice are not 3's is $\left(\frac{5}{6}\right)^5$.

(c) (5 points) What is the probability that at least one 3 appears?

This is the complement of part (b). So the answer is 1 minus the probability of getting no 3's, $1 - \left(\frac{5}{6}\right)^5$. You could also answer this by summing over probabilities of getting exactly one 3, exactly two 3's, exactly three 3's, and so on.

(d) (5 points) What is the probability that at most one 3 appears?

If at most one 3 appears, then either no 3's appear or exactly one 3 appears. So the answer is the sum of parts (a) and (b): $2 \cdot \left(\frac{5}{6}\right)^5$

(e) (5 points) What is the probability that no 3's appear, given that either 3 or 6 appears on each die?

Since every die shows a 3 or a 6, this is like asking, "What is the probability that no heads appear when five fair coins are flipped?" And the answer to that is $\left(\frac{1}{2}\right)^5$.

A longer, more formal way to look at it is to say E_1 is the event that no 3's appear and E_2 is the event that each die shows either 3 or 6. Then $p(E_2) = \left(\frac{2}{6}\right)^5$, and $p(E_1 \cap E_2) = \left(\frac{1}{6}\right)^5$ (this is the probability that all five dice come up 6). So the conditional probability is

$$p(E_1|E_2) = \frac{p(E_1 \cap E_2)}{p(E_2)} = \frac{\left(\frac{1}{6}\right)^5}{\left(\frac{2}{6}\right)^5} = \left(\frac{1}{2}\right)^5$$

4. Random Variables and Expected Values (9 points total)

Suppose you are walking past a broken ATM, when it suddenly starts spitting out money. Assume that the probability that a particular type of bill comes out is as follows:

Denomination	Probability
$10	0.65
$20	0.20
$100	0.15

(a) (5 points) If you take only one bill and walk away, what is the expected amount you will get?

The set of possible outcomes is $S = \{\$10, \$20, \$100\}$. Let $X(s)$ be a random variable that is equal to the amount of each outcome s. Then the expected value of one bill is

$$\begin{aligned} \sum_{s \in S} p(s) \cdot X(s) &= (\$10 \times 0.65) + (\$20 \times 0.20) + (\$100 \times 0.15) \\ &= \$6.50 + \$4.00 + \$15.00 \\ &= \$25.50 \end{aligned}$$

It's important to understand that the *expected* value can be a fraction.

(b) (4 points) Suppose the ATM spits out a total of 100 bills before it stops. What is the expected total amount that the ATM spits out? (Assume that the value of each bill that is spit out is an independent random variable.)

The expected total amount is just the sum of the expected amounts for each bill:

$$100 \cdot \$25.50 = \$2550.00$$

5. Sampling with Repetition (15 points total)

   (a) (5 points) How many different combinations of pennies, nickels, dimes, and quarters can a piggy bank contain if it has 20 coins in it?

   This is equivalent to distributing twenty indistinguishable objects into four distinct bins. The number of ways to do that is

$$\binom{20+4-1}{20} = \frac{23!}{3!\,20!} = 1771$$

   You could also equate this to having twenty stars and three bars, where the stars to the left of the first bar represent pennies, the stars between the first and second bar represent nickes, and so on.

   (b) (5 points) How many strings can be made from the letters in *MISSISSIPPI*, using all the letters?

   There are eleven letters, including 1 M, 4 I's, 4 S's, and 2 P's. So the number of distinct strings is

$$\frac{11!}{1!\,4!\,4!\,2!} = 11 \cdot 10 \cdot 9 \cdot 7 \cdot 5 = 34,650$$

   This could also be solved by choosing 1 of 11 places for the M, 4 of remaining 10 places for I's, 4 of remaining 6 for S's, then 2 of just 2 remaining for P's.

$$\binom{11}{1} \cdot \binom{10}{4} \cdot \binom{6}{4} \cdot \binom{2}{2} = \left(\frac{11!}{1!\,10!}\right)\left(\frac{10!}{6!\,4!}\right)\left(\frac{6!}{4!\,2!}\right)\left(\frac{2!}{2!\,0!}\right) = \frac{11!}{1!\,4!\,4!\,2!}$$

   (c) (5 points) How many different solutions are there to the equation:

$$v + w + x + y + z = 10$$

   where v, x, y, and z are nonnegative integers (i.e. ≥ 0) and w is an integer greater than zero (i.e. $w > 0$)?

If w were allowed to equal zero, then this would be like distributing 10 indistinguishable objects into five distinct bins. But since w has to be at least 1, let's define a new variable $w' = w - 1$. Then the above equation can be rewritten as

$$v + w' + x + y + z = 9$$

where *all* the variables are ≥ 0. So the number of solutions is

$$\binom{9+5-1}{9} = \frac{13!}{9!\,4!} = 13 \cdot 11 \cdot 5 = 715$$

6. Recurrence Relations (14 points)

(a) (7 points) Find the solution to the recurrence

$$a_n = 9a_{n-1} - 20a_{n-2} + 12$$

where $a_0 = 1$ and $a_1 = 2$.

First solve the corresponding homogeneous recurrence $a_n = 9a_{n-1} - 20a_{n-2}$. The characteristic equation is $r^2 - 9r + 20 = (r-5)(r-4) = 0$, so the roots are 4 and 5. And solutions to the homogeneous equation will have the form $a_n^{(h)} = \alpha_1 \cdot 5^n + \alpha_2 \cdot 4^n$.

Now we need a particular solution. It wouldn't be too hard to guess one just by looking at the recurrence. But if you don't want to guess, you could use Theorem 6 from section 5.2 of Rosen. According to the theorem, the inhomogeneous part is $F(n) = 12$, which has the form $b_0 s^n$, where $b_0 = 12$ and $s = 1$ (and $t = 0$). Since s is not a root of the homogeneous equation, our solution will look like $a_n^{(p)} = p_0 s^n = p_0$, which is just a constant. Plugging that back into the recurrence gives $p_0 = 9p_0 - 20p_0 + 12$. Solving for p_0 gives us a particular solution $a_n^{(p)} = 1$.

A general solution will look like $a_n^{(h)} + a_n^{(p)} = \alpha_1 \cdot 5^n + \alpha_2 \cdot 4^n + 1$. From our initial conditions, we have $a_0 = 1 = \alpha_1 + \alpha_2 + 1$ and $a_1 = 2 = 5\alpha_1 + 4\alpha_2 + 1$. Solving for the constants gives us $\alpha_1 = 1$ and $\alpha_2 = -1$. So the answer is $a_n = 5^n - 4^n + 1$.

(b) (7 points) Find the solution to the recurrence

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$$

where $a_0 = 1$, $a_1 = 6$, and $a_2 = 15$.

The characteristic equation is $r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0$. (You should be able to recognize this from Pascal's triangle.) So 1 is a root with multiplicity 3. The solution will have the form $a_n = \alpha_1 n^2 1^n + \alpha_2 n 1^n + \alpha_3 1^n = \alpha_1 n^2 + \alpha_2 n + \alpha_3$. The initial conditions give us this system of equations:

$$
\begin{aligned}
a_0 &= 1 &= \alpha_3 \\
a_1 &= 6 &= \alpha_1 + \alpha_2 + \alpha_3 \\
a_2 &= 15 &= 4\alpha_1 + 2\alpha_2 + \alpha_3
\end{aligned}
$$

Solving this system gives $\alpha_1 = 2$, $\alpha_2 = 3$, and $\alpha_3 = 1$. So the final answer is

$$a_n = 2n^2 + 3n + 1$$