Motion Planning for Dynamic Environments

Part I - Motion Planning: Living in C-Space

Steven M. LaValle
University of Illinois
Given obstacles, a robot, and its motion capabilities, compute collision-free robot motions from the start to goal.
Geometric Models

Transforming Robots

Topology

C-Spaces

Metric Spaces

C-Space Obstacles
The robot and obstacles live in a \textit{world} or \textit{workspace} $\mathcal{W}$. Usually, $\mathcal{W} = \mathbb{R}^2$ or $\mathcal{W} = \mathbb{R}^3$.

The \textit{obstacle region} $\mathcal{O} \subset \mathcal{W}$ is a closed set. The \textit{robot} $\mathcal{A}(q) \subseteq \mathcal{W}$ is a closed set.

(placed at configuration $q$).

Representation issues:

- Can it be obtained automatically or with little processing?
- What is the complexity of the representation?
- Can collision queries be efficiently resolved?
- Can a solid or surface be easily inferred?
Geometric Models: Linear Primitives

\[ f(x, y) = ax + by + c \]

Inside: \( f(x, y) \leq 0 \)

Intersections make convex polygons or polyhedra.

Notions of inside and outside are clear.
Consider primitives of the form:

\[ H_i = \{(x, y, z) \in \mathcal{W} \mid f_i(x, y, z) \leq 0\}, \]

which is a *half-space* is \( f_i \) is linear.

Now let \( f_i \) be any polynomial, such as \( f(x, y) = x^2 + y^2 - 1 \).

Obstacles can be formed from finite intersections:

\[ \mathcal{O} = H_1 \cap H_2 \cap H_3 \cap H_4. \]

And from finite unions of those:

\[ \mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_n. \]

\( \mathcal{O} \) could then become any *semi-algebraic* set.
In CAD models inside-outside may not be clearly defined.

Throw it all into a collision checker and hope for the best...

A typical representation: Triangle strips and fans
The most natural: Take data straight from range sensors

See the Point Cloud Library.

Problem: Hard to define and test for “collision”
Geometric Models

Transforming Robots

Topology

C-Spaces

Metric Spaces

C-Space Obstacles

Transforming Robots
May be rigid, articulated, deformable, reconfigurable, ...

The *degrees of freedom* is important.
Consider $\mathcal{W} = \mathbb{R}^2$ and $\mathcal{A} \subset \mathbb{R}^2$.

**Translation:**
Translate $\mathcal{A}$ by $x_t \in \mathbb{R}$ and $y_t \in \mathbb{R}$.
This means for every $(x, y) \in \mathcal{A}$, we obtain

$$(x, y) \mapsto (x + x_t, y + y_t)$$

The result is denoted as $\mathcal{A}(x_t, y_t)$. 
Rotation: Rotate $\mathcal{A}$ by $\theta \in [0, 2\pi)$

This means for every $(x, y) \in \mathcal{A}$, we obtain

$$(x, y) \mapsto (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$$

The result is $\mathcal{A}(\theta)$. 
Combining Translation and Rotation

Important: Rotate first, then translate

$$(x, y) \mapsto \begin{pmatrix} x \cos \theta - y \sin \theta + x_t \\ x \sin \theta + y \cos \theta + y_t \end{pmatrix}$$

The operations can be performed by a matrix:

$$\begin{pmatrix} \cos \theta & -\sin \theta & x_t \\ \sin \theta & \cos \theta & y_t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x \cos \theta - y \sin \theta + x_t \\ x \sin \theta + y \cos \theta + y_t \end{pmatrix}$$

Technically: A rigid body transformation is an orientation-preserving, isometric embedding.
The 3 by 3 matrix

\[
T(x_t, y_t, \theta) = \begin{pmatrix}
\cos \theta & -\sin \theta & x_t \\
\sin \theta & \cos \theta & y_t \\
0 & 0 & 1
\end{pmatrix}
\]

contains a rotation matrix in the upper left and a translation column vector on the right.

\[
T(x_t, y_t, \theta) = \begin{pmatrix}
R(\theta) & v \\
0 & 1
\end{pmatrix}
\]

in which

\[
R(\theta) = \begin{pmatrix}
x \cos \theta - y \sin \theta \\
x \sin \theta + y \cos \theta
\end{pmatrix}
\]

and \( v = (x, y, y_t) \).
Now, \( \mathcal{W} = \mathbb{R}^3 \) and \( \mathcal{A} \subset \mathbb{R}^3 \).

**Translation:**

Translate \( \mathcal{A} \) by \( x_t, y_t, z_t \in \mathbb{R} \).

This means for every \( (x, y) \in \mathcal{A} \), we obtain

\[
(x, y) \mapsto (x + x_t, y + y_t, z + z_t)
\]

The result is denoted as \( \mathcal{A}(x_t, y_t, z_t) \).
Rotation:

Yaw: Rotation of $\alpha$ about the $z$-axis:

$$R_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Pitch: Rotation of $\beta$ about the $y$-axis:

$$R_y(\beta) = \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}.$$  

Roll: Rotation of $\gamma$ about the $x$-axis:

$$R_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & -\sin \gamma \\ 0 & \sin \gamma & \cos \gamma \end{pmatrix}.$$
Combining them is sufficient to produce any rotation:

\[ R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_x(\gamma) = \]

\[
\begin{pmatrix}
\cos \alpha \cos \beta & \cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \cos \gamma + \sin \alpha \sin \gamma \\
\sin \alpha \cos \beta & \sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \cos \gamma - \cos \alpha \sin \gamma \\
-\sin \beta & \cos \beta \sin \gamma & \cos \beta \cos \gamma
\end{pmatrix}.
\]

Every rotation matrix must have:

- Unit column vectors
- Pairwise orthogonal columns
- Determine 1
We now obtain a 4 by 4 homogeneous transformation matrix:

\[ T(\alpha, \beta, x_t, y_t, z_t) = \begin{pmatrix} R(\alpha, \beta, \gamma) & v \\ 0 & 1 \end{pmatrix}. \]
For $n$ independent bodies, just use $n$ separate homogeneous transformation matrices.

However, if they are non-rigidly attached:

then use specialized, chained transformations.
One matrix for each link:

\[
T_1 = \begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 & x_t \\
\sin \theta_1 & \cos \theta_1 & y_t \\
0 & 0 & 1
\end{pmatrix}
\]

A chain of matrices for the chain of links:

\[
T_1 T_2 \cdots T_m \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}
\]
In three dimensions, bodies may be non-rigidly attached in many ways:

- **Revolute**: 1 Degree of Freedom
- **Prismatic**: 1 Degree of Freedom
- **Screw**: 1 Degree of Freedom
- **Cylindrical**: 2 Degrees of Freedom
- **Spherical**: 3 Degrees of Freedom
- **Planar**: 3 Degrees of Freedom

Nevertheless, systems of parametrizations are developed: Denavit-Hartenburg, Khalil-Kleinfinger, ...
General idea: Need to find good parametrizations of the freedom of motion between attached links.

Warning: Extremely hard for closed chains.
Topology
Path planning becomes a search on a space of transformations

What does this space look like?

How should it be represented?

What alternative representations are allowed and how do they affect performance?
Three views of the configuration space:

1. As a topological manifold
2. As a metric space
3. As a differentiable manifold

Number 3 is too complicated! There is no calculus in basic path planning.
Start with any set $X$.

Declare some of the sets in $\text{pow}(X)$ to be \textit{open} sets. If these hold:

1. The union of \textbf{any number} of open sets is an open set.

2. The intersection of a \textbf{finite number} of open sets is an open set.

3. Both $X$ and $\emptyset$ are open sets.

then $X$ is a \textit{topological space}.

A set $C \subseteq X$ is \textit{closed} if and only if $X \setminus C$ is open.

Many subsets of $X$ could be neither open nor closed.
Although elegant, the previous definition was much too general.

We will only consider spaces of the form $X \subseteq \mathbb{R}^n$. 

$\mathbb{R}^n$ comes equipped with standard open sets: 

A set $O$ is open if every $x \in O$ is contained in a ball that is contained in $O$.

To get the open sets of $X$, take every open set $O \subseteq \mathbb{R}^n$ and form $O' = O \cap X$. 
With respect to a subset $U \subseteq X$, a point $x \in X$ may be:

- a *boundary point*, as in $x_1$ above,
- an *interior point*, as in $x_2$,
- or an *exterior point*, as in $x_3$. 

![Diagram showing a subset $U$ with points $x_1$, $x_2$, $x_3$, and obstacles $O_1$ and $O_2$.](Image)
Let $X$ and $Y$ be any topological spaces.

A function $f : X \rightarrow Y$ is called \textit{continuous} if for any open set $O \subseteq Y$, the preimage $f^{-1}(O) \subseteq X$ is an open set.
A bijection \( f : X \to Y \) is called a **homeomorphism** if both \( f \) and \( f^{-1} \) are continuous.

If \( f \) exists, then \( X \) and \( Y \) are **homeomorphic**.

Example: For \( X = (-1, 1) \) and \( Y = \mathbb{R} \), let \( x \mapsto 2 \tan^{-1}(x)/\pi \) \((-1, 1)\).

These are all homeomorphic subspaces of \( \mathbb{R}^2 \).

These are homeomorphic, but not with the ones above them.
These are all mutually non-homeomorphic
Let $M \subseteq \mathbb{R}^m$ be any set that becomes a topological space using the subset topology.

$M$ is called a *manifold* if for every $x \in M$, an open set $O \subset M$ exists such that: 1) $x \in O$, 2) $O$ is homeomorphic to $\mathbb{R}^n$, and 3) $n$ is fixed for all $x \in M$.

It “feels like” $\mathbb{R}^n$ around every $x \in M$. 
Subspaces of $\mathbb{R}^2$:

- Yes
- Yes
- Yes
- No

- Yes
- No
- Yes
- No

All it takes is one bad point to fail the manifold test.
$\mathbb{R}^n$ is a distinct manifold for each $n$

$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is a circle manifold

Here are some 2D cylinders (all homeomorphic!):

Another one: $M = \mathbb{R}^2 \setminus \{(0, 0)\}$ (the punctured plane)
Start with an open square \((0, 1)^2 \subset \mathbb{R}^2\)

Let \((x, y)\) denote a point on the manifold.

Include the \(x = 0\) points and define equivalence relation \(\sim\):

\[(0, y) \sim (1, y)\]

for all \(y \in (0, 1)\).
Typical appearance

Change the equivalence relation to

\[(0, y) \sim (1, 1 - y)\]

for all \(y \in (0, 1)\).
Many useful, distinct manifolds can be made by identifying edges of a polytope.

<table>
<thead>
<tr>
<th>Plane, $\mathbb{R}^2$</th>
<th>Cylinder, $\mathbb{R} \times S^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Möbius band</td>
<td>Torus</td>
</tr>
<tr>
<td>Klein bottle</td>
<td>Projective plane, $\mathbb{R}P^2$</td>
</tr>
<tr>
<td>Two-sphere, $S^2$</td>
<td>Double torus</td>
</tr>
</tbody>
</table>
C-Spaces
A simple way to describe the manifold of all transformations

\[ T(q) = \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix} \]

\( SE(n) \) is the group of all \((n + 1)\) by \((n + 1)\) dimensional homogeneous transformation matrices.

Thus, \( SE(2) \) is just a subset of \( \mathbb{R}^9 \) and \( SE(3) \) is a subset of \( \mathbb{R}^{16} \). But which matrices are allowed? Is there a nice parametrization?
The **configuration space** $\mathcal{C}$ is the set of all allowable robot transformations.

Translation parameters: $x_t, y_t \in \mathbb{R}$

Rotation parameter: $\theta \in [0, 2\pi]$

Using the homeomorphism $\theta \mapsto (\cos \theta, \sin \theta)$, the space of all rotations is $S^1$.

The configuration space is $\mathcal{C} = \mathbb{R}^2 \times S^1$.

Note “=” here means “homeomorphic to”
Recall that \( \mathbb{R} \times S^1 \) is a cylinder. 
\[ C = \mathbb{R}^2 \times S^1 \] can be imagined as a “thick” cylinder.

Or a square box with the top and bottom identified:
Translation parameters: $x_t, y_t, z_t \in \mathbb{R}$ Rotation parameters: yaw, pitch, roll?

Gimbal lock problem: An infinite number of YPR parameters map to the same rotation.

When the pitch is $90^\circ$, yaw and roll become the same.
(First roll, then pitch, then yaw)
Consider the mapping:

\[(a, b, c, d) \mapsto \begin{pmatrix}
2(a^2 + b^2) - 1 & 2(bc - ad) & 2(bd + ac) \\
2(bc + ad) & 2(a^2 + c^2) - 1 & 2(cd - ab) \\
2(bd - ac) & 2(cd + ab) & 2(a^2 + d^2) - 1
\end{pmatrix}\]

in which \(a, b, c, d \in \mathbb{R}\).

Enforce the constraint \(a^2 + b^2 + c^2 + d^2 = 1\).

In this case, the mapping above is two-to-one everywhere onto \(SO(3)\). \((a, b, c, d)\) and \((-a, -b, -c, -d)\) map to the same rotation.
These are the same rotation.

If you like algebra, consider $(a, b, c, d)$ as a quaternion.
Use upper half of $S^3$: $d \geq 0$ and $a^2 + b^2 + c^2 + d^2 = 1$

Project down: $(a, b, c, d) \mapsto (a, b, c, 0)$.

The result is a 3D ball: $B_3 = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 \leq 1\}$.

However, on the boundary of $B_3$ we have $(a, b, c) \sim (-a, -b, -c)$. 
Stretching $B_3$ out to make a cubes.

Opposite faces are reverse identified; hence, $B_3 = \mathbb{RP}^3$.

Alternatively, could stretch $S^3$ out to the faces of the 4-cube. The 4-cube as 8 faces, but only 4 cubes are needed.
For a rigid body that translates and rotates in $\mathbb{R}^3$:

$$C = \mathbb{R}^3 \times \mathbb{RP}^3$$

The $\mathbb{R}^3$ components arise from translation.
The $\mathbb{RP}^3$ component arises from rotation.
For independent bodies, \( A_1 \) and \( A_2 \), take the Cartesian product:

\[
C = C_1 \times C_2
\]

If they are attached to make a kinematic chain, then take the Cartesian product of their components:

\[
C = \mathbb{R}^2 \times S^1 \times S^1 \times S^1
\]
The case of closed kinematic chains often arises in redundant robots, manipulation, protein folding, ...

A manifold may result, but it may be difficult to obtain an efficient parametrization.
Convenient parametrizations preferred

Geometric distortion should be minimized

How should be distortion be described? Metric space.
Metric Spaces
A *metric space* $(X, \rho)$ is a topological space $X$ equipped with a function \( \rho : X \times X \to \mathbb{R} \) such that for any $a, b, c \in X$:

1. **Nonnegativity**: \( \rho(a, b) \geq 0 \).
2. **Reflexivity**: \( \rho(a, b) = 0 \) if and only if $a = b$.
3. **Symmetry**: \( \rho(a, b) = \rho(b, a) \).
4. **Triangle inequality**: \( \rho(a, b) + \rho(b, c) \geq \rho(a, c) \).

Example: Euclidean distance in $\mathbb{R}^n$

More examples: $L_p$ metrics in $\mathbb{R}^n$
Map onto a unit circle, and then use Euclidean distance:

$$p_1 = (\cos \theta_1, \sin \theta_1) \quad \text{and} \quad p_2 = (\cos \theta_2, \sin \theta_2)$$

Direct comparison of angles in $\mathbb{R}$:

$$\rho(\theta_1, \theta_2) = \min \{ |\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2| \}$$

or

$$\rho(a_1, b_1, a_2, b_2) = \cos^{-1}(a_1 a_2 + b_1 b_2),$$

in which $a_i = \cos \theta_i$ and $b_i = \sin \theta_i$. 
Comparing rotations in $SO(3)$ works in a similar way, using the $h = (a, b, c, d)$ representation:

$$\rho_s(h_1, h_2) = \cos^{-1}(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)$$ \hspace{1cm} (1)

However, must consider identification of antipodal points:

$$\rho(h_1, h_2) = \min \{ \rho_s(h_1, h_2), \rho_s(h_1, -h_2) \}.$$ \hspace{1cm} (2)

Other possibilities: Euclidean distance in yaw-pitch-roll space, Euclidean distance in $\mathbb{R}^9$ (the space of 3 by 3 matrices).

Some metrics are more “natural” than others. How to formalize?
Let $G$ be a matrix group, such as $SO(n)$ or $SE(n)$.

Let $\mu$ be a measure on $G$. In could, for example, assign volumes by using the metric function.

If for any measurable subset $A \subseteq G$, and any element $g \in G$, $\mu(A) = \mu(gA) = \mu(Ag)$, then $\mu$ is called the Haar measure. The Haar measure exists for any locally compact topological group and is unique up to scale.

Example for $SO(2)$ using the unit circle $S^1$:
For 3D rotations, recall the mapping

\[(a, b, c, d) \mapsto SO(3)\]  

The Haar measure for \(SO(3)\) is obtained as the standard area (or 3D volume) on the surface of \(S^3\).

Uniform random points on \(S^3\) yield uniform random rotations on \(SO(3)\) that are compatible with the Haar measure (it is the right way to sample).
Let \((X, \rho_x)\) and \((Y, \rho_y)\) be two metric spaces. A metric space for the Cartesian product \(Z = X \times Y\) is formed as

\[
\rho_z(z, z') = \rho_z(x, y, x', y') = c_1 \rho_x(x, x') + c_2 \rho_y(y, y'),
\]

in which \(c_1, c_2\) are positive constants.

If \(X = \mathbb{R}^2\) from translation and \(Y = S^1\) from rotation, what should \(c_1\) and \(c_2\) be?

Perhaps \(c_2 = c_1/r\), in which \(r\) is the point on \(A\) that is furthest from the origin.

What should the constants be for a long kinematic chain?
C-Space Obstacles
Given world $W$, a closed obstacle region $O \subset W$, closed robot $A$, and configuration space $C$.
Let $A(q) \subset W$ denote the placement of the robot into configuration $q$.

The **obstacle region** $C_{obs}$ in $C$ is

$$ C_{obs} = \{ q \in C \mid A(q) \cap O \neq \emptyset \}, $$

which is a closed set.

The **free space** $C_{free}$ is an open subset of $C$:

$$ C_{free} = C \setminus C_{obs} $$

We want to keep the configuration in $C_{free}$ at all times!
Consider $C_{obs}$ for the case of translation only.

The Minkowski sum of two sets is defined as

$$X \oplus Y = \{x + y \in \mathbb{R}^n \mid x \in X \text{ and } y \in Y\}$$

(from the CGAL manual)
The Minkowski difference of two sets is defined as

\[ X \ominus Y = \{ x - y \in \mathbb{R}^n \mid x \in X \text{ and } y \in Y \} \]  

(6)

A one-dimensional example:

\[ -4 -3 -2 -1 0 1 2 3 4 5 6 \]

\[ \overline{A} \quad \overline{O} \quad \overline{A} \quad \overline{C_{\text{obs}}} \]

Sometimes called convolution.
The C-Space Obstacle
A simple algorithm to compute the obstacle.

Inward and outward normals

Sorted around $S^1$

$C_{obs}$

$O$
The C-Space Obstacle

<table>
<thead>
<tr>
<th>Type</th>
<th>Vtx.</th>
<th>Edge</th>
<th>$n$</th>
<th>$v$</th>
<th>Half-Plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>VE</td>
<td>$a_3$</td>
<td>$b_4$-$b_1$</td>
<td>$[1, 0]$</td>
<td>$[x_t - 2, y_t]$</td>
<td>${ q \in \mathbb{C} \mid x_t - 2 \leq 0 }$</td>
</tr>
<tr>
<td>VE</td>
<td>$a_3$</td>
<td>$b_1$-$b_2$</td>
<td>$[0, 1]$</td>
<td>$[x_t - 2, y_t - 2]$</td>
<td>${ q \in \mathbb{C} \mid y_t - 2 \leq 0 }$</td>
</tr>
<tr>
<td>EV</td>
<td>$b_2$</td>
<td>$a_3$-$a_1$</td>
<td>$[1, 2]$</td>
<td>$[-x_t, 2 - y_t]$</td>
<td>${ q \in \mathbb{C} \mid -x_t + 2y_t - 4 \leq 0 }$</td>
</tr>
<tr>
<td>VE</td>
<td>$a_1$</td>
<td>$b_2$-$b_3$</td>
<td>$[-1, 0]$</td>
<td>$[2 + x_t, y_t - 1]$</td>
<td>${ q \in \mathbb{C} \mid -x_t - 2 \leq 0 }$</td>
</tr>
<tr>
<td>EV</td>
<td>$b_3$</td>
<td>$a_1$-$a_2$</td>
<td>$[1, 1]$</td>
<td>$[-1 - x_t, -y_t]$</td>
<td>${ q \in \mathbb{C} \mid -x_t - y_t - 1 \leq 0 }$</td>
</tr>
<tr>
<td>VE</td>
<td>$a_2$</td>
<td>$b_3$-$b_4$</td>
<td>$[0, -1]$</td>
<td>$[x_t + 1, y_t + 2]$</td>
<td>${ q \in \mathbb{C} \mid -y_t - 2 \leq 0 }$</td>
</tr>
<tr>
<td>EV</td>
<td>$b_4$</td>
<td>$a_2$-$a_3$</td>
<td>$[-2, 1]$</td>
<td>$[2 - x_t, -y_t]$</td>
<td>${ q \in \mathbb{C} \mid 2x_t - y_t - 4 \leq 0 }$</td>
</tr>
</tbody>
</table>
What about translation and rotation?

Obtain a 3D subset of $\mathbb{R}^2 \times S^1$.

Two contact types:

Equations polynomial in $x_t, y_t, a, b$ arise.

$(a = \cos \theta \text{ and } b = \sin \theta)$

Forms the boundary of a 3D semi-algebraic obstacle in $C = \mathbb{R}^2 \times S^1$
In 3D, there are three contact types:

1. **Type FV**: A face of $\mathcal{A}$ and a vertex of $\mathcal{O}$
2. **Type VF**: A vertex of $\mathcal{A}$ and a face of $\mathcal{O}$
3. **Type EE**: An edge of $\mathcal{A}$ and an edge of $\mathcal{O}$

Forms the boundary of a 6D semi-algebraic obstacle in $C = \mathbb{R}^3 \times \mathbb{RP}^3$
The Obstacles in C-Space Can Be Complicated

For the case of two-links, $C = S^1 \times S^1$, but the obstacle region can quickly become strange and complicated:
Given robot $A$ and obstacle $O$ models, C-space $C$, and $q_I, q_G \in C_{free}$.

Automatically compute a path $\tau : [0, 1] \rightarrow C_{free}$ so that $\tau(0) = q_I$ and $\tau(1) = q_G$. 
Geometric representations are an important first step.

Planning is a search on the space of transformations.

Think like a topologist when it comes to C-space.

A car driving on a gigantic sphere:

The sphere is large enough so that the car does not wobble.

The car can achieve any position and orientation on the sphere.

What is the C-space?