Abstract Sensing Models

- We want a kind of “cartoon-like” description of the universe

- A state is a description of the universe (at a particular time) that is sufficient for determining the observation (output) of an ideal time-invariant sensor.
Ex: Distance sensor

\[ \mathbb{R}^2 \]

\[ (a, b) \]

\[ \theta \]

Given \( y \)

\[ \{ x \in X \mid y = h(x) \} \]

\[ y = h(a, b, \theta) \]

\[ h: X \to Y \]

\( X \): state space = set of all possible \((a, b, \theta)\)

\( Y \): observation space = set of all possible distances
**Example: Detecting people**

$Y = \{0, 1\}$

- $0$: beam is not broken
- $1$: beam broken

What should $X$ be?

Consider coordinates for each person: $(a_i, b_i)$

$X = (a_1, b_1, a_2, b_2, \ldots, a_5, b_5)$

$X \subseteq \mathbb{R}^{10}$

We can write:

$y = h(x)$

- $y = 1$ if $(a_i, b_i) \in B$ for any $i \in \{1, \ldots, 5\}$
- $y = 0$ otherwise
At a high level, we have two fundamental choices for the environment model:

1) 2D or 3D
2) Continuous or discrete

For many sensor models, we need to define:

- A configuration space
- Environment boundary (obstruction to motion or sensors)
Configuration Spaces

Lagrange (1772-1788)

See my book, Chapter 4 (and 3.2, 3.3)

Key idea: For any movable body, or collection of bodies, describe their position & orientation using only one parameter for each degree of freedom.

Consider the case of 2D environment

Suppose the body is a point in $\mathbb{R}^2$

Assume no obstructions
Let $C$ denote the configuration space $C = \mathbb{R}^2$.

Replace the point by a rigid body.

- Rigid body
- Define it in the body frame

Apply a translation $(x_t, y_t)$ to move every $p = (x, y) \in A$ to $(x + x_t, y + y_t)$.

The configuration space remains $C = \mathbb{R}^2$.

Now consider rotation of the body.

$SO(2)$ is the set of all rotation matrices (lavalle, 4.2.1) (group).
SO(2) can be parameterized nicely as

\[ \{ R(\theta) \mid \theta \in [0, 2\pi) \} \]

in which

\[ R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

Technically, \( C = SO(2) \)

Alternatively, we would like \( C = [0, 2\pi) \)

\[ \text{HPOEVER, note that } R(0) = R(2\pi) \]

Thus, 0 and 2\pi are equivalent

Can use identification to declare 0 and 2\pi to be the same.

\( C = [0, 2\pi] \sim \)

\[ \begin{array}{c}
  0 \\
  2\pi \\
  0 \end{array} \]

\( 0 \sim 2\pi \) equivalence relation
Alternative parameterization of $SO(2)$

$\{ R(a, b) \mid a^2 + b^2 = 1 \}$

in which $R(a, b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$SO(2)$ looks like a circle

Let $S'$ denote the unit circle at the origin of $\mathbb{R}^2$

Note that $a = \cos \theta, b = \sin \theta$

Thus, by a simple transformation, $\mathbb{R}^2 \sim [0, 2\pi]$ and $S'$

seem to be the "same" space and $SO(2)$

In topology, this is made rigorous: homomorphism (LeValle, 4.2.4.3)

$C = S' \quad C = SO(2) \quad C \sim [0, 2\pi]$
Now allow the body to rotate and translate.

From translation, we had \( C = \mathbb{R}^2 \)
From rotation, we had \( C = S^1 \)

Set of all translations and rotations is \( SE(2) \)

All 3x3 matrices of the form:

\[
\begin{bmatrix}
a & -b & x_t \\
b & a & y_t \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a^2 + b^2 = 1 \\
x_t, y_t \in \mathbb{R}
\end{bmatrix}
\]

Use **Cartesian product**

Two spaces, \( X \) and \( Y \), \( x \in X, y \in Y \)

Cartesian product \( X \times Y \) is a new space.

For which every element is \((x,y)\), an ordered pair, such that \( x \in X, y \in Y \)

**Ex**:

\( X = \mathbb{R}, Y = \mathbb{R} \quad X \times Y = \mathbb{R}^2 \)
\( X = \mathbb{R}, Y = S^1 \quad X \times Y = \mathbb{R} \times S^1 \)
Translation: $C = \mathbb{R}^2$
Rotation: $C = S^1$

Let $H: C = \mathbb{R}^2 \times S^1$

$\mathbb{R}^2 \times S^1$

Note: We can even transform a point using translation and rotation.

Orientation is important for the sensor model.
Some variations:
1) multiple moving bodies
2) discrete models
3) 3D environments
4) obstructions
5) Attached bodies

1) multiple moving bodies
Take the Cartesian product, combining each body's configuration space.

Ex. 3 points that translate (only)

\[(x_1, y_1), (x_2, y_2), (x_3, y_3)\]

State
\[(x_1, y_1, x_2, y_2, x_3, y_3)\]

\[C = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^6\]
Ex Two rigid bodies that rotate and translate

\[ C = (\mathbb{R}^2 \times S^1) \times (\mathbb{R}^2 \times S^1) = \mathbb{R}^4 \times S^1 \times S^1 \]

\[ (x_1, y_1, \theta_1, x_2, y_2, \theta_2) \in C \]

\[ C \text{ is six dimensional} \]

\[ S^2 = \text{sphere} \]

\[ S^1 \times S^1 \neq S^2 \]

\[ S^1 \times S^1 \]

\[ \text{torus} \]
2) Discrete models

Consider a grid

Positions are discretized, so $\mathbb{R}^2$ becomes $\mathbb{Z} \times \mathbb{Z}$, set of all integers

Translation can be written as $(i, j) \in \mathbb{Z} \times \mathbb{Z}$

Rotations are discretized into 4 values

$\{0, \pi/2, \pi, 3\pi/2\} = D

\{N, E, W, S\}$

The discretized configuration space

$C = \mathbb{Z} \times \mathbb{Z} \times D$
3) 3D environments

Define a 3D rigid body, A, in a body frame \( A \subset \mathbb{R}^3 \)

Translation is \( (x_t, y_t, z_t) \) and moves each point to \( (x + x_t, y + y_t, z + z_t) \)

Translation only: \( C = \mathbb{R}^3 \)

Rotation is more complicated

\( SO(3) \) is the set of all 3x3 rotation matrices

(LeVelle, 4.2.2)

Recall for \( SO(2) \) we used \((a,b)\) and \((a, -b)\) to get \( C = S^1 \)
SO(2) \rightarrow (a, b) \rightarrow \text{complex } a + ib

SU(3) \rightarrow h = (a, b, c, d) \rightarrow \text{quaternion } a + bi + cj + d

\begin{align*}
R(h) &= 2 \begin{pmatrix}
(a^2 + b^2)^{-1} & (bc - ad) & (bd + ac) \\
(bc + ad) & (a^2 + c^2)^{-1} & (cd - ab) \\
b^2 - ac & (cd + ab) & (a^2 + d^2)^{-1}
\end{pmatrix} \\
\text{SO(3)} &= \{ R(h) \mid a^2 + b^2 + c^2 + d^2 = 1 \}
\end{align*}

\text{Could have } C = S^3 \quad 3 \text{ dimensional sphere in } \mathbb{R}^4

\text{However, } \quad h = (a, b, c, d) \text{ and } -h = (-a, -b, -c, -d)

\text{yield the same rotation.}

\text{For } SE(3) - \text{rotation } & \text{translation}

\text{For } C^* = \mathbb{RP}^3 - \text{real projective space}

\text{rotation only}

\text{a} \geq 0

\mathbb{RP}^2 \quad 6 \text{ dimensional}

\mathbb{R}^3 \times \mathbb{RP}^3

\mathbb{S}^2 \text{ rotation only}
- planet (perfect sphere)
- huge (car doesn't wobble)
- what is the configuration space of the car?

\[ C = S^2 \times S^1 \]

Next

4) obstructions
5) attached bodies

more general state spaces