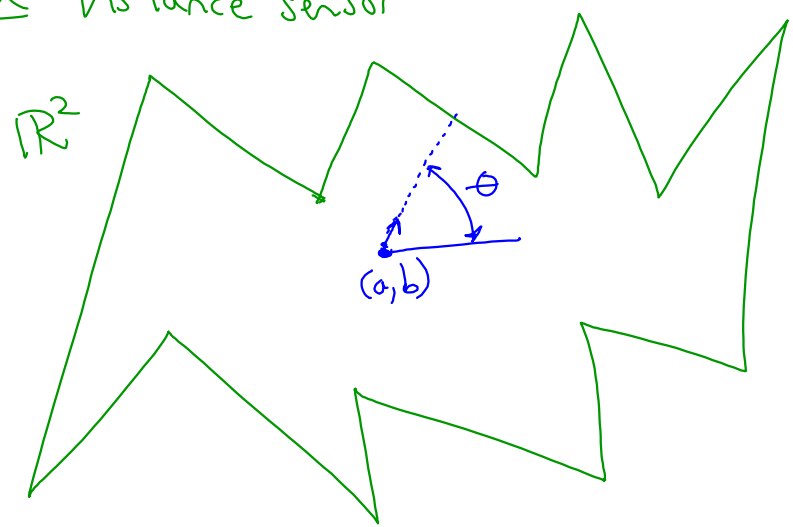
Abstract Sensing Models

- We want a kind of "cartoon-like" description of the universe
- A <u>state</u> is a description of the universe (at a particular time) that is sufficient for determining the observation (output) of an ideal time-invariant sensor.

Ex Distance sensor



$y = h(a,b,\theta)$

$h: X \to Y$

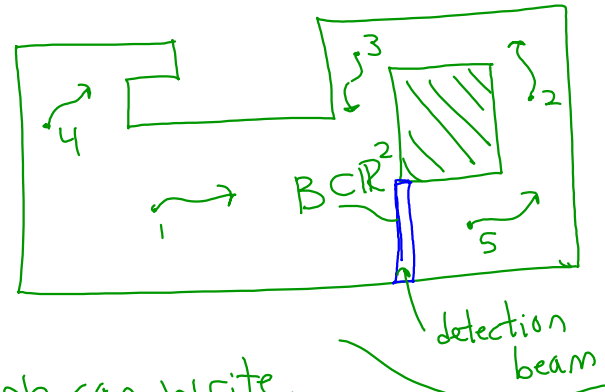
Given y

$\{x \in X \mid y = h(x)\}$

X = state space = set of all possible (a,b,θ)

Y = observation space = set of all possible distances

Ex Detecting people



$Y = \{0, 1\}$ 0 = beam is not broken
1 = beam broken

What should X be?

Consider coords for each person: (a_i, b_i)

$x = (a_1, b_1, a_2, b_2, \ldots, a_5, b_5)$ $X \subset \mathbb{R}^{10}$

We can write

$y = h(x)$ $y = 1$ if $(a_i, b_i) \in B$ for any $i \in \{1, \ldots, 5\}$
$y = 0$ otherwise

At a high level, we have two fundamental choices for the environment (universe) model

1) 2D or 3D
2) continuous or discrete

For many sensor models, we need to define:

- A configuration space
- Environment boundary (obstruction to motion or sensors)

Configuration Spaces Lagrange (1772-1788)

See my book, Chapter 4 (and 3.2, 3.3)

Key idea: For any movable body, or collection of bodies, describe their position & orientation using only one parameter for each degree of freedom.

Consider the case of 2D environment

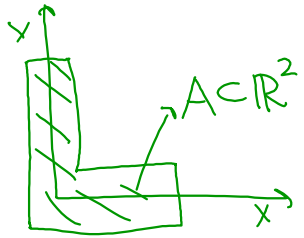
Suppose the body is a point in $\mathbb{R}^2$

Assume no obstructions

Let $\mathcal{C}$ denote the <u>configuration space</u>

$\mathcal{C} = \mathbb{R}^2$

Replace the point by a rigid body.



- Rigid body
- Define it in the <u>body frame</u>

Apply a <u>translation</u> (x_t, y_t) to move every $p = (x, y) \in A$ to $(x + x_t, y + y_t)$

<u>The</u> configuration space remains $\mathcal{C} = \mathbb{R}^2$

Now consider <u>rotation</u> of the body

$SO(2)$ is the set of all rotation matrices (Lavalle, 4.2.1)
(group)

$SO(2)$ can be parameterized nicely as

$$\{R(\theta) \mid \theta \in [0, 2\pi)\}$$

in which $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Technically, $\mathcal{C} = SO(2)$

Alternatively, we would like $\mathcal{C} = [0, 2\pi)$

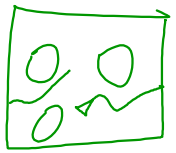
HOWEVER, note that $R(0) = R(2\pi)$

Thus, 0 and 2π are equivalent

Can use <u>identification</u> to declare 0 and 2π to be the same.

$0 \sim 2\pi$ equivalence relation

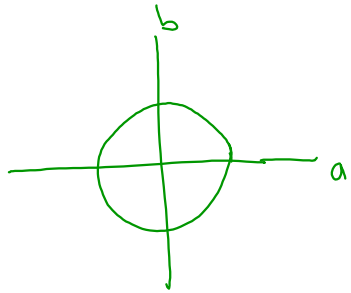
$\mathcal{C} = [0, 2\pi] \backslash \sim$



Alternative parameterization of $SO(2)$

$$\{ R(a,b) \mid a^2+b^2=1 \}$$

in which $R(a,b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$



$SO(2)$ looks like a circle

Let S^1 denote the unit circle at the origin of $\mathbb{R}^2$

Note that $a = \cos\theta$ $b = \sin\theta$

Thus, by a simple transformation $[0,2\pi]\backslash\sim$ and S^1 seem to be the "same" space and $SO(2)$

In topology, this is made rigorous: homeomorphism (LaValle, 4.2.4.3)

$\mathcal{C} = S^1$ $\mathcal{C} = SO(2)$ $\mathcal{C} = [0,2\pi]\backslash\sim$

Now allow the body to rotate and translate

From translation, we had $\mathcal{C} = \mathbb{R}^2$

From rotation, we had $\mathcal{C} = S^1$

Set of all translations and rotations is $SE(2)$

All 3x3 matrices of the form

$$\left\{ \underbrace{\begin{bmatrix} a & -b & x_t \\ b & a & y_t \\ 0 & 0 & 1 \end{bmatrix}}_{T(a,b,x_t,y_t)} \;\middle|\; \begin{array}{l} a^2+b^2=1 \\ x_t, y_t \in \mathbb{R} \end{array} \right\}$$

$$T(a,b,x_t,y_t)\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Use <u>Cartesian product</u>

Two spaces, X and Y, $x \in X$, $y \in Y$

Cartesian product $X \times Y$ is a new space.

for which every element is (x,y), an ordered pair, such that $x \in X$, $y \in Y$

<u>Ex</u> $X = \mathbb{R}$ $Y = \mathbb{R}$ $X \times Y = \mathbb{R}^2$

$X = \mathbb{R}$ $Y = S^1$ $X \times Y = \mathbb{R} \times S^1$

Translation: $C = \mathbb{R}^2$

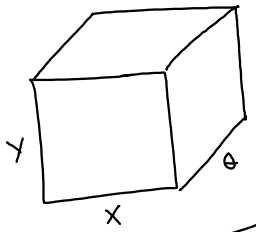
Rotation: $C = S^1$

Both: $\boxed{C = \mathbb{R}^2 \times S^1}$

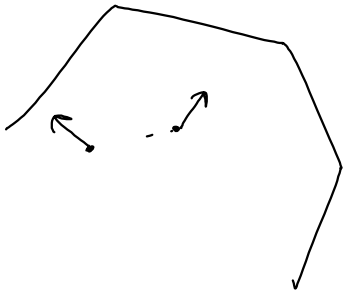
$\mathbb{R} \times S^1$

$[0,1] \times [0,1] \backslash \sim$

$\mathbb{R}^2 \times S^1$



Note: we can even transform a point using translation and rotation.



Orientation is important for the sensor model.

Some variations:

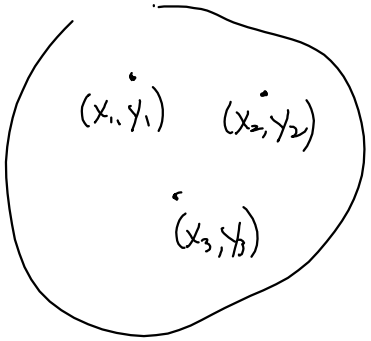
1) multiple moving bodies
2) discrete models
3) 3D environments
4) obstructions
5) Attached bodies

1) multiple moving bodies

Take the Cartesian product, combining each body's configuration space.

<u>Ex</u> 3 points that translate (only)

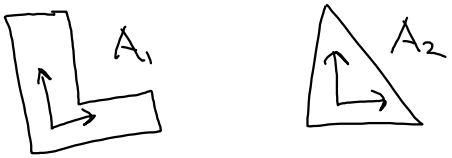
(x_1, y_1) (x_2, y_2) (x_3, y_3)



state $(x_1, y_1, x_2, y_2, x_3, y_3)$

$$\mathcal{C} = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^6$$

<u>Ex</u> Two rigid bodies that rotate and translate



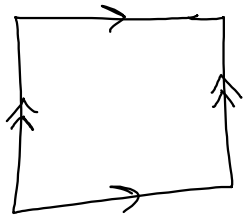
$$\mathcal{C} = (\mathbb{R}^2 \times S^1) \times (\mathbb{R}^2 \times S^1) = \mathbb{R}^4 \times S^1 \times S^1$$

$$(x_1, y_1, \theta_1, x_2, y_2, \theta_2) \in \mathcal{C}$$

$\mathcal{C}$ is six dimensional

S^2 = sphere

$S^1 \times S^1 \neq S^2$

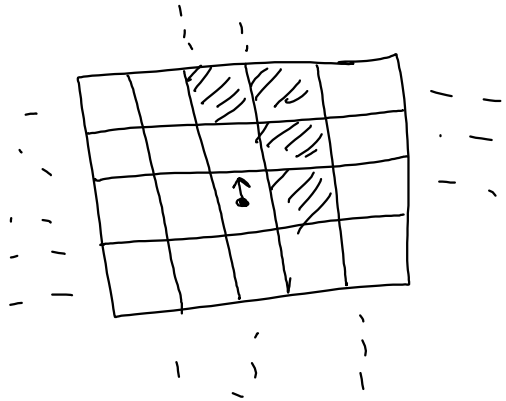


$S^1 \times S^1$

torus

2) Discrete models

Consider a grid



Positions are discretized,

so $\mathbb{R}^2$ becomes $\mathbb{Z} \times \mathbb{Z}$ ← set of all integers

Translation can be written as $(i,j) \in \mathbb{Z} \times \mathbb{Z}$

Rotations are discretized into 4 values

$\{0, \pi/2, \pi, 3\pi/2\} = D$

$\{N, E, W, S\}$

The discretized configuration space

$C = \mathbb{Z} \times \mathbb{Z} \times D$

3) 3D environments

Define a 3D rigid body, A, in a body frame. $A \subset \mathbb{R}^3$

<u>Translation</u> is (x_t, y_t, z_t) and moves each <u>point</u> to $(x+x_t, y+y_t, z+z_t)$

$(x,y,z) \in A$

Translation only: $\mathcal{C} = \mathbb{R}^3$

<u>Rotation</u> is more complicated

$SO(3)$ is the set of all 3×3 rotation matrices (LaValle, 4.2.2)

Recall for $SO(2)$ we used (a,b) and $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ to get $\mathcal{C} = S^1$

$SO(2) \to (a,b) \longrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \to$ complex $ai+b$

$SO(3) \to h=(a,b,c,d) \longrightarrow$ quaternion $ai+bj+ck+d$

$$R(h) = 2\begin{pmatrix} (a^2+b^2)-1 & (bc-ad) & (bd+ac) \\ (bc+ad) & (a^2+c^2)-1 & (cd-ab) \\ (bd-ac) & (cd+ab) & (a^2+d^2)-1 \end{pmatrix}$$

$$SO(3) = \{ R(h) \mid \underline{a^2+b^2+c^2+d^2=1} \}$$

Could have $\mathcal{C} = S^3$ — 3 dimensional sphere in $\mathbb{R}^4$

However,

$h=(a,b,c,d)$ and $-h=(-a,-b,-c,-d)$

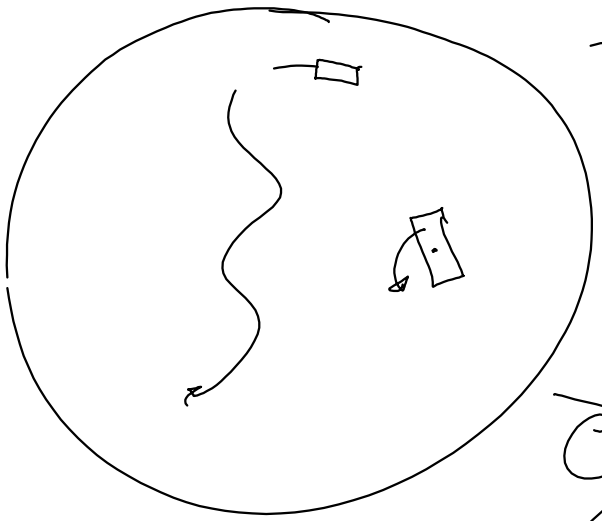
yield the same rotation.

$\mathcal{C} = \mathbb{R}P^3$ — real projective space — rotation only

For $SE(3)$ - rotation & translation

$\mathcal{C} = \mathbb{R}^3 \times \mathbb{R}P^3$ — 6 dimensional

$a \geq 0$



- planet (perfect sphere)
- huge (car doesn't wobble)
- what is the configuration space of the car?

$\cancel{C = S^2 \times S^1}$

Next

4) obstructions
5) attached bodies

more general state spaces